An Unofficial Guide to Math 16020
Spring 2019

Ellen Weld

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Course Information

1. Course Calendar: Spring 2019

	Date	**Lesson**	**Assignment/Topic**
	1/7 M	R	Review of Basic Integration
	1/9 W	1A	Integration by Substitution (I)
	1/11 F	1B	Integration by Substitution (II)
	1/14 M	2	Integration by Substitution (III)
	1/16 W	3	The Natural Logarithm Function: Integration
	1/18 F	4	Integration by Parts (I)
No Class	1/21 M	—	Martin Luther King Jr. Day
	1/23 W	5	Integration by Parts (II)
	1/25 F	6	Differential Equations: Solutions, Growth, and Decay
Exam 1	1/28 M	R-5	**Exam 1: 8:00 PM, ELLIOT 116 (No Class)**
	1/30 W	7	Differential Equations: Separation of Variables (I)
	2/1 F	8	Differential Equations: Separation of Variables (II)
	2/4 M	9	First-Order Linear Differential Equations (I)
	2/6 W	10	First-Order Linear Differential Equations (II)
	2/8 F	11	Area of a Region between Two Curves
	2/11 M	12	Volume of Solids of Revolution (I)
	2/13 W	13	Volume of Solids of Revolution (II)
	2/15 F	14	Volume of Solids of Revolution (III)
	2/18 M	15	Improper Integrals
	2/20 W	16	Geometric Series and Convergence (I)
	2/22 F	17	Geometric Series and Convergence (II)
Exam 2	2/25 M	6-16	**Exam 2: 8:00 PM, ELLIOT 116 (No Class)**
	2/27 W	18	Functions of Several Variables Intro
	3/1 F	19	Partial Derivatives (I)
	3/4 M	20	Partial Derivatives (II)
	3/6 W	21	Differentials of Multivariable Functions
	3/8 F	22	Chain Rule, Functions of Several Variables

No Class	3/11 M	—	Spring Break
No Class	3/13 W	—	Spring Break
No Class	3/15 F	—	Spring Break
	3/18 M	23	Extrema of Functions of Two Variables (I)
	3/20 W	24	Extrema of Functions of Two Variables (II)
	3/22 F	25	LaGrange Multipliers - Constrained Min/Max (I)
	3/25 M	26	LaGrange Multipliers - Constrainted Min/Max (II)
	3/26 T	16-25	**Exam 3: 8:00 PM ELLIOT 116**
No Class	3/27 W	—	**No Class**
	3/29 F	27	Double Integrals, Volume, Applications (I)
	4/1 M	28	Double Integrals, Volume, Applications (II)
	4/3 W	29	Double Integrals, Volume, Applications (III)
	4/5 F	30	Systems of Equations, Matrices, Gaussian Elimination
	4/8 M	31	Gauss-Jordan Elimination
	4/10 W	32	Matrix Operations
	4/12 F	33	Inverses and Determinants of Matrices (I)
Exam 4	4/15 M	25-32	**Exam 4: 8:00 PM, ELLIOT 116 (No Class)**
	4/18 W	34	Inverses and Determinants of Matrices (II)
	4/20 F	35	Eigenvalues and Eigenvectors (I)
	4/22 M	36	Eigenvalues and Eigenvectors (II)
	4/24 W	Review	**Review for Final Exam**
	4/26 F	Review	**Review for Final Exam**
Finals	4/29-5/4	—	**Finals Week**

2. Basic Information

General Information	
Webpage	www.math.purdue.edu/MA16020
Grades	On Blackboard

Course Coordinator: Owen Davis	
Email	davisok@purdue.edu

Instructor: Ellen Weld	
Email	weld@purdue.edu
Webpage	www.math.purdue.edu/∼ weld/
Office	Math 639

3. Course Policies

Please be sure to read the syllabus on the course webpage for more general course policies.

Calculators: You are allowed a 1-line scientific calculator on all quizzes and exams. Check the course webpage for a graphic detailing what a 1-line scientific calculator is.

Homework: Homework on each lesson is due the morning before the next lesson. All homework is done on Loncapa, an online homework system developed by Michigan State University with problems programmed by Purdue graduate students. You will have 10 tries on each homework problem unless otherwise noted.

All homework questions must be directed to Piazza (linked through Blackboard). I will not answer homework questions in my email.

Exams: There will be 4 midterm exams and a final exam. The midterm exams will be 75 minutes long in Elliot Hall 116 and the final will be 2 hours long.

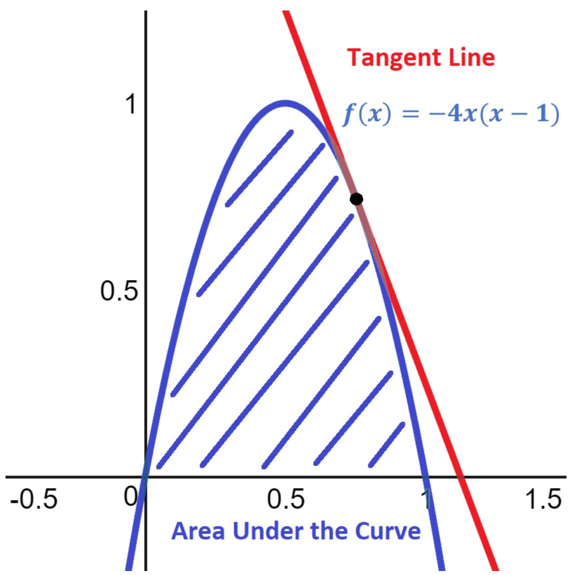
Quizzes: Quizzes are graded out of 10 points where turning in *anything* with your name on it will garner 1 point. The only way to get 0 points is to not turn in anything.

Textbook: The **official** course textbook is found on Loncapa between the homework questions. It includes videos and written examples. Take time to watch the videos and read through the examples because many times they will directly address homework problems. **This document is not the course textbook**.

Lesson R: Review of Basic Integration

1. Differentiation versus Integration

Differentiation		Integration
$\frac{d}{dx}$, f' Tangent lines How a Function Changes	$\overset{FTC}{\longleftrightarrow}$	$\int f(x)\,dx$, $\int_a^b f(x)\,dx$ Area under a curve How a Function Accumulates



Theorem 1 (**Fundamental Theorem of Calculus (FTC)**).

(a) $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$

(b) $\displaystyle\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

Fact 2. The Fundamental Theorem of Calculus (FTC) tells us that derivatives and integrals are "opposites", i.e., differentiation undoes integration and integration undoes differentiation.

Integrals come in two flavors: indefinite and definite.

The **indefinite integral** of $f(x)$, denoted $\int f(x)\,dx$, is the **list** of all functions $F(x)$ that $F'(x) = f(x)$. All such $F(x)$ differ by a constant, which is where the $+C$ comes from.

A **definite integral** is a **number** that describes the area under the curve over a specified interval. A definite integral always includes bounds. This may be represented by a graph.

Ex 1. The definite integral of x^2 from -1 to 1 is represented by this graph

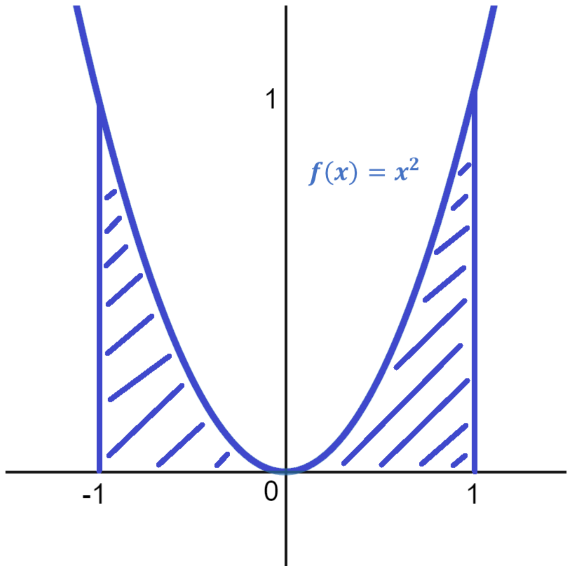


Figure 1. This is the geometric representation of $\int_{-1}^{1} x^2\,dx$.

For a definite integral, you need bounds. Here, the bounds are $x = -1$ and $x = 1$ (which corresponds to the interval $[-1, 1]$).

2. Basic Integration

Examples.

1. Evaluate $\int 7x^3\,dx$ and $\int_1^2 7x^3\,dx$.

Solution: The first integral is an *indefinite* integral and the second is a *definite* integral. We write

$$\begin{aligned}\int 7x^3\,dx &= 7\int x^3\,dx\\ &= 7\left(\frac{1}{3+1}x^{3+1}\right) + C\\ &= 7\left(\frac{1}{4}\right)x^4 + C\\ &= \boxed{\frac{7}{4}x^4 + C}\end{aligned}$$

Next, we find the definite integral:

$$\begin{aligned}\int_1^2 7x^3\,dx &= 7\int_1^2 x^3\,dx\\ &= 7\left(\frac{1}{3+1}\right)x^{3+1}\Big|_1^2\\ &= 7\left(\frac{1}{4}\right)x^4\Big|_1^2\\ &= \frac{7}{4}x^4\Big|_1^2\\ &= \frac{7}{4}\left[(2)^4-(1)^4\right]\\ &= \frac{7}{4}\left[16-1\right]\\ &= \frac{7}{4}(15)\\ &= \boxed{\frac{105}{4}}\end{aligned}$$

Observe that we didn't need to do the full integration again to find the definite integral. Besides adding a $+C$, the function we find in the indefinite integral is the function at which we evaluate the bounds for the definite integral.

2. Evaluate $\int 2e^x\,dx$ and $\int_0^{\ln 2} 2e^x\,dx$.

Solution: Write

$$\begin{aligned}\int 2e^x\,dx &= 2\int e^x\,dx\\ &= 2(e^x)+C\\ &= \boxed{2e^x+C}\end{aligned}$$

Next,

$$\begin{aligned}\int_0^{\ln 2} 2e^x\,dx &= 2e^x\Big|_0^{\ln 2}\\ &= 2\underbrace{e^{\ln 2}}_{\diamond}-2e^0\\ &= 2(2)-2(1)\end{aligned}$$

$\diamond$ $e^{\ln x}=x$, hence $e^{\ln 2}=2$

$$= 4 - 2$$

$$= \boxed{2}$$

3. Evaluate $\displaystyle\int \frac{1}{2}\sec^2 x\,dx$ and $\displaystyle\int_0^{\pi/3} \frac{1}{2}\sec^2 x\,dx$.

Solution: Trig functions will periodically show up in this class although there is no lesson specifically addressing them. Take time to review these functions for the handful of times they appear.

$$\begin{aligned}\int \frac{1}{2}\sec^2 x\,dx &= \frac{1}{2}\int \sec^2 x\,dx \\ &= \boxed{\frac{1}{2}\tan x + C} \\ \int_0^{\pi/3} \frac{1}{2}\sec^2 x\,dx &= \frac{1}{2}\int_0^{\pi/3} \sec^2 x\,dx \\ &= \frac{1}{2}\tan x\Big|_0^{\pi/3} \\ &= \frac{1}{2}\tan(\pi/3) - \frac{1}{2}\tan(0) \\ &= \frac{1}{2}\sqrt{3} - \frac{1}{2}(0) \\ &= \boxed{\frac{\sqrt{3}}{2}}\end{aligned}$$

4. Evaluate $\displaystyle\int \frac{3}{x}\,dx$ and $\displaystyle\int_e^{e^7} \frac{3}{x}\,dx$.

Solution: Recall that

$$\frac{1}{x} = x^{-1}$$

and this is the single exception to the power rule:

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Write

$$\begin{aligned}\int \frac{3}{x}\,dx &= 3\int \frac{1}{x}\,dx \\ &= \boxed{3\ln|x| + C}\end{aligned}$$

Next,

$$\begin{aligned}
\int_e^{e^7} \frac{3}{x}\,dx &= 3\int_e^{e^7} \frac{1}{x}\,dx \\
&= 3\ln|x|\Big|_e^{e^7} \\
&= 3\ln|e^7| - 3\ln|e| \\
&= 3\underbrace{\ln(e^7)}_{\diamond\diamond} - 3\underbrace{\ln(e)}_{\diamond\diamond} \text{ since } e, e^7 > 0 \\
&= 3(7) - 3(1) \\
&= 21 - 3 \\
&= \boxed{18}
\end{aligned}$$

5. Evaluate $\displaystyle\int_0^6 (4e^x - 9)\,dx$.

Solution: We are allowed to split up an integral over a plus or minus sign. Write

$$\begin{aligned}
\int_0^6 (4e^x - 9)\,dx &= \int_0^6 4e^x\,dx - \int_0^6 9\,dx \\
&= 4\int_0^6 e^x\,dx - \int_0^6 9\,dx \\
&= 4e^x\Big|_0^6 - 9x\Big|_0^6 \\
&= 4e^x - 9x\Big|_0^6 \\
&= 4e^6 - 9(6) - [4\underbrace{e^0}_{1} - 9(0)] \\
&= 4e^6 - 54 - [4 - 0] \\
&= 4e^6 - 54 - 4 \\
&= \boxed{4e^6 - 58}
\end{aligned}$$

6. Find $\displaystyle\int (x-1)^2\,dx$.

Solution: Before we integrate, we need to FOIL out the function. Write

$$(x-1)^2 = (x-1)(x-1)$$

$\diamond\diamond$ $\ln e^x = x$, hence $\ln e^7 = 7, \ln e = \ln e^1 = 1, \ln 1 = \ln e^0 = 0$

$$= x(x) + (-1)(x) + (-1)(x) + (-1)(-1)$$
$$= x^2 - x - x + 1$$
$$= x^2 - 2x + 1$$

REMARK 3. Observe that $(x-1)^2 \neq x^2 + 1$.

Write

$$\begin{aligned}
\int (x-1)^2\,dx &= \int (x-1)(x-1)\,dx \\
&= \int (x^2 - 2x + 1)\,dx \\
&= \int x^2\,dx + \int (-2x)\,dx + \int 1\,dx \\
&= \frac{1}{2+1}x^{2+1} - \frac{2}{1+1}x^{1+1} + x + C \\
&= \frac{1}{3}x^3 - \frac{2}{2}x^2 + x + C \\
&= \boxed{\frac{1}{3}x^3 - x^2 + x + C}
\end{aligned}$$

7. Find $\displaystyle\int_1^4 \frac{x^4 + \sqrt{x^3}}{\sqrt{x}}\,dx$.

Solution: We need to remember how to deal with exponents. A square root is denoted by a $\frac{1}{2}$ in the exponent, so we write

$$\sqrt{x^3} = (x^3)^{1/2} = x^{3/2}.$$

Hence, our function can be rewritten as

$$\frac{x^4 + \sqrt{x^3}}{\sqrt{x}} = \frac{x^4 + x^{3/2}}{x^{1/2}}.$$

We can further simplify our function:

$$\begin{aligned}
\frac{x^4 + x^{3/2}}{x^{1/2}} &= \frac{x^4}{x^{1/2}} + \frac{x^{3/2}}{x^{1/2}} \\
&= x^{4-1/2} + x^{3/2-1/2} \\
&= x^{8/2-1/2} + x^{3/2-1/2} \\
&= x^{7/2} + x^{2/2} \\
&= x^{7/2} + x^1
\end{aligned}$$

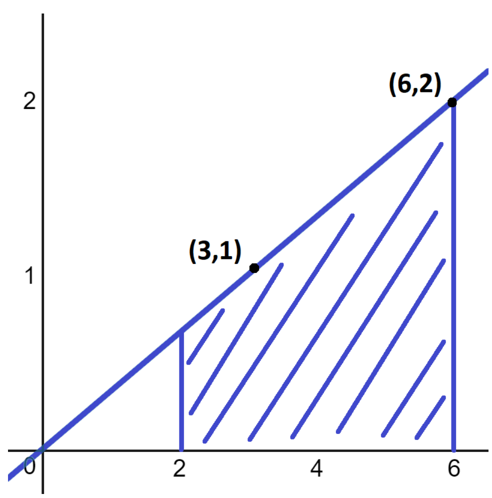
Now, we can go forward with our integration:

$$\begin{aligned}
\int_1^4 \frac{x^4+\sqrt{x^3}}{\sqrt{x}}\,dx &= \int_1^4 \frac{x^4+x^{3/2}}{x^{1/2}}\,dx \\
&= \int_1^4 \left(x^{7/2}+x\right)\,dx \\
&= \frac{1}{7/2+1}x^{7/2+1} + \frac{1}{1+1}x^{1+1}\bigg|_1^4 \\
&= \frac{1}{7/2+2/2}x^{7/2+2/2} + \frac{1}{2}x^2\bigg|_1^4 \\
&= \frac{1}{9/2}x^{9/2} + \frac{1}{2}x^2\bigg|_1^4 \\
&= \frac{2}{9}x^{9/2} + \frac{1}{2}x^2\bigg|_1^4 \\
&= \frac{2}{9}(4)^{9/2} + \frac{1}{2}(4)^2 - \left[\frac{2}{9}(1)^{9/2} + \frac{1}{2}(1)^2\right] \\
&= \frac{2}{9}(4^{1/2})^9 + \frac{1}{2}(16) - \left[\frac{2}{9} + \frac{1}{2}\right] \\
&= \frac{2}{9}(2)^9 + 8 - \frac{2}{9} - \frac{1}{2} \\
&= \boxed{\frac{2179}{18}}
\end{aligned}$$

3. Additional Examples

EXAMPLES.

1. Find the definite integral that is described by the following:



Solution: We need two things: **(1)** the function and **(2)** the bounds.

(1) This line passes through $(3, 1)$ and $(6, 2)$. Thus, its slope is given by $\dfrac{2-1}{6-3} = \dfrac{1}{3}$. Further, the line passes through the point $(0, 0)$. Recall that point slope form is given by

$$y - y_0 = m(x - x_0)$$

which means the function of this line is

$$y - 0 = \frac{1}{3}(x - 0) \quad \Rightarrow \quad y = \frac{1}{3}x.$$

(2) The bounds are $x = 2$ and $x = 6$ because the shaded region lies between these x-values.

Thus, the definite integral that is represented by the graph above is

$$\boxed{\int_2^6 \frac{1}{3}x\,dx}.$$

2. What is the value of the definite integral from # 2?

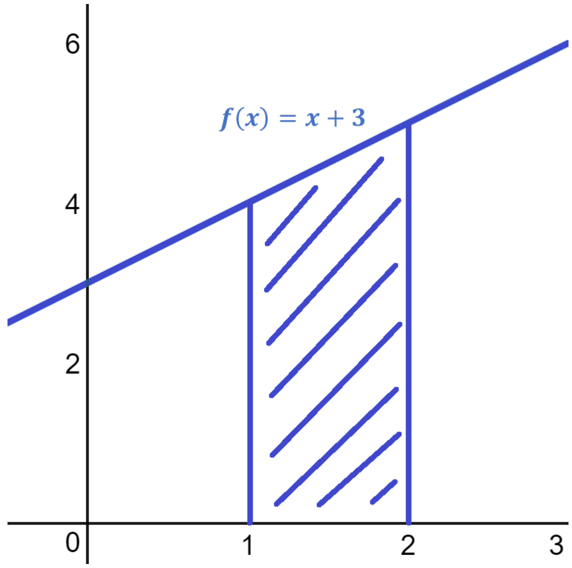
Solution: We write

$$\begin{aligned}
\int_2^6 \frac{1}{3}x\,dx &= \int_2^6 \frac{1}{3}x^1\,dx \\
&= \left(\frac{1/3}{1+1}x^{1+1}\right)\Bigg|_2^6 \\
&= \frac{1/3}{2}x^2\Bigg|_2^6 \\
&= \frac{1}{6}x^2\Bigg|_2^6 \\
&= \frac{1}{6}(6)^2 - \frac{1}{6}(2)^2 \\
&= \frac{36}{6} - \frac{4}{6} \\
&= \frac{32}{6} = \boxed{\frac{16}{3}}
\end{aligned}$$

3. Find the area of the region bounded by

$$y = x + 3, \quad y = 0, \quad x = 1, \quad x = 2.$$

Solution: First, draw the graph.



Second, set up the definite integral. We want to find the area below $x+3$ and between $x=1$ and $x=2$, which is just the integral

$$\int_1^2 (x+3)\,dx.$$

Finally, we compute this definite integral:

$$\begin{aligned}\int_1^2 (x+3)\,dx &= \int_1^2 x\,dx + \int_1^2 3\,dx \\ &= \frac{1}{1+1}x^{1+1}\Big|_1^2 + 3x\Big|_1^2 \\ &= \frac{1}{2}x^2 + 3x\Big|_1^2 \\ &= \frac{1}{2}(2)^2 + 3(2) - \left(\frac{1}{2}(1)^2 + 3(1)\right) \\ &= \frac{1}{2}(4) + 6 - \frac{1}{2} - 3 \\ &= 2 + 6 - \frac{1}{2} - 3 \\ &= \boxed{\frac{9}{2}}.\end{aligned}$$

4. Suppose you start driving on the highway at 10:00 AM at a speed given by $s(t) = \frac{15}{2}t + 45$ mile/hour.

(a) How far have you gone by 12:30PM? Round your answer to the nearest hundredth.

Solution: Observe that this problem asks for a rounded answer, **not** an *exact* answer. Check Appendix D for an example of the difference between exact and rounded answers.

We need to find the distance traveled, that is, the number of miles the car has accumulated, after 2.5 hours. But accumulation is what integrals are built to compute. So, after we write down our definite integral, we need only integrate and we'll have our answer. The integral we are looking for is given by

$$\int_0^{2.5} \left(\frac{15}{2}t + 45\right)\, dt.$$

Next, write

$$\begin{aligned}
\int_0^{2.5} \left(\frac{15}{2}t + 45\right)\, dt &= \frac{15}{2}\left(\frac{1}{2}\right)t^2 + 45t\Big|_0^{2.5} \\
&= \frac{15}{4}t^2 + 45t\Big|_0^{2.5} \\
&= \frac{15}{4}(2.5)^2 + 45(2.5) - \left[\frac{15}{4}(0)^2 - 45(0)\right] \\
&= \frac{2175}{16} - 0 \\
&\approx \boxed{135.94 \text{ miles}}.
\end{aligned}$$

(b) After how many hours will you have gone 100 miles? Round your answer to the nearest hundredth.

Solution: In the previous question, we were asked about the distance traveled after a certain time. Here, we are given the *distance* and asked about the *time*.

By our work in the previous problem, we have

$$\text{distance traveled} = \int_0^{\text{time traveled}} \left(\frac{15}{2}t + 45\right)\, dt.$$

We are given distance traveled $= 100$. So, if we let $x =$ time traveled, then we get this equation:

$$100 = \int_0^{x} \left(\frac{15}{2}t + 45\right)\, dt.$$

Thus, our goal is to find x.

Write

$$100 = \int_0^{x} \left(\frac{15}{2}t + 45\right)\, dt = \frac{15}{4}x^2 + 45x.$$

Subtracting 100 from both sides, we see we need to solve

$$\frac{15}{4}x^2 + 45x - 100 = 0.$$

After an application of the quadratic formula, we get

$$x \approx \boxed{1.92 \text{ hours}}.$$

5. Evaluate

$$\int_{-\pi/6}^{\pi/6} \cot x \sin x \, dx$$

Solution: We observe that since

$$\cot x = \frac{\cos x}{\sin x},$$

we may simplify this function:

$$\cot x \sin x = \frac{\cos x}{\sin x}(\sin x) = \cos x.$$

Thus,

$$\begin{aligned}\int_{-\pi/6}^{\pi/6} \cot x \sin x \, dx &= \int_{-\pi/6}^{\pi/6} \cos x \, dx \\ &= \sin x \Big|_{-\pi/6}^{\pi/6} \\ &= \sin(\pi/6) - \sin(-\pi/6) \\ &= \frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2} = \boxed{1}\end{aligned}$$

6. Find $y(2)$ if $y' = x^2$ and $y(1) = -1$.

Solution: First, we find a generic antiderivative of y' and, second, we use the initial condition $y(1) = -1$ to solve for C. Once we have the specific function y, we evaluate at $x = 2$.

So,

$$\begin{aligned}\int x^2 \, dx &= \frac{1}{2+1}x^{2+1} + C \\ &= \frac{1}{3}x^3 + C.\end{aligned}$$

Next, since $y(1) = -1$, we may write

$$\begin{aligned} -1 &= \frac{1}{3}(1)^3 + C \\ &= \frac{1}{3} + C \\ \Rightarrow \quad -\frac{4}{3} &= C \end{aligned}$$

Hence, the specific function is

$$y = \frac{1}{3}x^3 - \frac{4}{3}.$$

Finally, we have

$$y(2) = \frac{1}{3}(2)^3 - \frac{4}{3} = \frac{8}{3} - \frac{4}{3} = \boxed{\frac{4}{3}}.$$

Lesson 1A: Integration by Substitution (I)

1. Integration by Substitution

Integration and differentiation are opposite actions. For example,

$$\int \boxed{x^{-1/2}}\, dx = \frac{1}{-1/2+1} x^{-1/2+1} + C = \frac{1}{1/2} x^{1/2} + C = \underline{2x^{1/2} + C}$$

and

$$\frac{d}{dx}(\underline{2x^{1/2} + C}) = 2\left(\frac{1}{2}\right) x^{1/2-1} = \boxed{x^{-1/2}}.$$

Now, suppose we are given the function $12x^2(x^3+7)^3$ and asked to integrate. If we knew that, by the chain rule,

$$\frac{d}{dx}(x^3+7)^4 = 4(x^3+7)^3(3x^2) = 12x^2(x^3+7)^3, \tag{1}$$

then

$$\int 12x^2(x^3+7)^3\, dx = (x^3+7)^4 + C \tag{2}$$

by the reasoning as above. But how would we have determined (2) if we didn't already have (1)?

The process of integrating a function that's the product of the chain rule is called **u-substitution**. In some sense, this integration method is a sort of reverse engineering. Remember that, for the chain rule, you have an "inside" function and an "outside" function. The derivative of such a function is the product of the derivative of the inside function with the derivative of the outside function evaluated at the inside function.

Ex 1.

$$\frac{d}{dx}(x^3+7)^4 = \underbrace{4(x^3+7)^3}_{\substack{\text{derivative of outside} \\ \text{evaluated at inside}}} \overbrace{(3x^2)}^{\substack{\text{derivative of} \\ \text{inside}}}$$

Here, the inside function is x^3+7 and the outside function is x^4.

Now, to apply u-sub in this case, let $u = x^3+7$, the inside function. Then, if we differentiate, we see that

$$\frac{du}{dx} = 3x^2.$$

Next, we substitute. Write

$$\int 4(\underbrace{x^3+7}_{u})^3 \underbrace{(3x^2)}_{du/dx}\,dx = \int 4u^3\left(\frac{du}{dx}\right)dx$$
$$= \int 4u^3\,du$$

But this we know how to integrate. We have

$$\int 4u^3\,du = \frac{4}{3+1}u^{3+1} + C = \frac{4}{4}u^4 + C = u^4 + C.$$

However, we have not answered the question. We were asked about $\int 4(x^3+7)^3(3x^2)\,dx$, which is a function in terms of x and we **must** respond in terms of x.

Since $u = x^3 + 7$, we substitute again to get

$$\int 12x^2(x^3+7)^3\,dx = u^4 + C = (x^3+7)^4 + C.$$

Important Note: For u-sub, the chosen u **must** eliminate the original variable else you cannot continue to integrate.

Ex 2. Suppose we are asked to evaluate

$$\int (x+1)^{-1/2}\,dx.$$

The inside function is $u = x + 1$ but if we just write

$$\int (x+1)^{-1/2}\,dx = \int u^{-1/2}\,dx$$

then this is **wrong**. dx refers to the variable with which we are integrating and it is difficult to make sense of integrating a function of u with respect to x in this context. As a result, we **must completely eliminate** the original variable.

If $u = x + 1$, then $\frac{du}{dx} = 1$. So

$$\int (x+1)^{-1/2}\,dx = \int u^{-1/2}\cdot 1 \cdot\,dx$$
$$= \int u^{-1/2}\cdot\frac{du}{dx}\cdot dx$$
$$= \int u^{-1/2}\,du$$
$$= \frac{1}{-1/2+1}u^{-1/2+1} + C$$
$$= \frac{1}{1/2}u^{1/2} + C$$

$$= 2u^{1/2} + C$$

$$= 2(x+1)^{1/2} + C$$

REMARK 4. If you are unable to completely eliminate the original variable via your choice of u, then either this is the wrong choice of u or this integral cannot be addressed via u-substitution.

EXAMPLES.

1. Evaluate $\int x\sqrt{10-2x^2}\,dx$.

Solution: Here, the inside function is $u = 10 - 2x^2$ and $\frac{du}{dx} = -4x$. But there is no $-4x$ in this integral, so what do we do? Well, there are a couple of ways to address this and which method you use depends on what makes the most sense to you.

Method 1: Replace x

If $\frac{du}{dx} = -4x$, then $x = -\frac{1}{4}\frac{du}{dx}$ and we can write

$$\begin{aligned}
\int x\sqrt{10-2x^2}\,dx &= \int \underbrace{-\frac{1}{4}\frac{du}{dx}}_{x}\sqrt{u}\,dx \\
&= \int -\frac{1}{4}\sqrt{u}\,\frac{du}{dx}\,dx \\
&= \int -\frac{1}{4}\sqrt{u}\,du \\
&= -\frac{1}{4}\int \sqrt{u}\,du \\
&= -\frac{1}{4}\int u^{1/2}\,du \\
&= -\frac{1}{4}\left(\frac{1}{1/2+1}\right)u^{1/2+1} + C \\
&= -\frac{1}{4}\left(\frac{1}{3/2}\right)u^{3/2} + C \\
&= -\frac{1}{4}\left(\frac{2}{3}\right)u^{3/2} + C \\
&= -\frac{1}{6}u^{3/2} + C \\
&= \boxed{-\frac{1}{6}(10-2x^2)^{3/2} + C}
\end{aligned}$$

Method 2: Replace dx

We know that $\frac{du}{dx} = -4x$, solving for dx we see that

$$dx = -\frac{du}{4x}.$$

Substituting, we get

$$\begin{aligned}\int x\sqrt{10-2x^2}\,dx &= \int x\sqrt{u}\,\underbrace{\left(-\frac{du}{4x}\right)}_{dx} \\ &= \int -\frac{1}{4}\sqrt{u}\,du \\ &\vdots \\ &= \boxed{-\frac{1}{6}(10-2x^2)^{3/2}+C}\end{aligned}$$

as shown above.

2. Evaluate $\int 5x^2 e^{x^3}\,dx$.

Solution: The inside function is $u = x^3$ and so $\frac{du}{dx} = 3x^2$. Again, there are a couple of ways to approach this substitution. We can solve $\frac{du}{dx} = 3x^2$ for x^2 or we can solve for dx.

Method 1: Replace x^2

We know that $x^2 = \frac{1}{3}\frac{du}{dx}$. So we write

$$\begin{aligned}\int 5x^2 e^{x^3}\,dx &= \int 5\underbrace{\left(\frac{1}{3}\frac{du}{dx}\right)}_{x^2} e^u\,dx \\ &= \int \frac{5}{3}e^u\frac{du}{dx}\,dx \\ &= \int \frac{5}{3}e^u\,du \\ &= \frac{5}{3}e^u + C \\ &= \boxed{\frac{5}{3}e^{x^3}+C}\end{aligned}$$

Method 2: Replace dx

Solving for dx, we get $dx = \dfrac{du}{3x^2}$. Write

$$\int 5x^2 e^{x^3}\,dx = \int 5x^2 e^u \underbrace{\left(\frac{du}{3x^2}\right)}_{dx}$$

$$= \int \frac{5}{3} e^u\,du$$

$$\vdots$$

$$= \boxed{\frac{5}{3} e^{x^3} + C}$$

as shown above.

3. Evaluate $\displaystyle\int \frac{\cos(2x)}{\sin^3(2x)}\,dx$.

Solution: For this integral, it is very important that we choose the correct u. If we choose the wrong u, then we won't be able to completely eliminate x.

Take $u = \sin(2x)$, then $\dfrac{du}{dx} = 2\cos(2x)$. Again, we can approach this in different ways.

Method 1: Replace $\cos(2x)$

If $\dfrac{du}{dx} = 2\cos(2x)$, then $\cos(2x) = \dfrac{1}{2}\dfrac{du}{dx}$. Write

$$\int \frac{\cos(2x)}{\sin^3(2x)}\,dx = \int \cos(2x) \frac{1}{\sin^3(2x)}\,dx$$

$$= \int \underbrace{\left(\frac{1}{2}\frac{du}{dx}\right)}_{\cos(2x)} \frac{1}{u^3}\,dx$$

$$= \int \frac{1}{2}\frac{1}{u^3}\,du$$

$$= \frac{1}{2}\int \frac{1}{u^3}\,du$$

$$= \frac{1}{2}\int u^{-3}\,du$$

$$= \frac{1}{2}\left(\frac{1}{-3+1}\right) u^{-3+1} + C$$

$$= \frac{1}{2}\left(\frac{1}{-2}\right) u^{-2} + C$$

$$= -\frac{1}{4}u^{-2} + C$$
$$= \boxed{-\frac{1}{4}(\sin(2x))^{-2} + C}$$

Method 2: Replace dx

Since $\dfrac{du}{dx} = 2\cos(2x)$, we know that $dx = \dfrac{du}{2\cos(2x)}$. Hence,

$$\int \frac{\cos(2x)}{\sin^3(2x)}\,dx = \int \frac{\cos(2x)}{u^3}\underbrace{\left(\frac{du}{2\cos(2x)}\right)}_{dx}$$
$$= \int \frac{1}{2u^3}\,du$$
$$\vdots$$
$$= \boxed{-\frac{1}{4}(\sin(2x))^{-2} + C}$$

as we see above.

2. Additional Examples

EXAMPLES.

1. Evaluate $\displaystyle\int \frac{(3+\sqrt{x})^3}{\sqrt{x}}\,dx$.

Solution: Typically when integrating, it is easiest to rewrite roots in terms of their fractional exponents. In particular, the function here can be rewritten

$$\frac{(3+\sqrt{x})^3}{\sqrt{x}} = \frac{(3+x^{1/2})^3}{x^{1/2}}.$$

REMARK 5. Observe that $(3+\sqrt{x})^3 \neq 3^3 + (\sqrt{x})^3$.

Written like this, take $u = 3 + x^{1/2}$, then $\dfrac{du}{dx} = \dfrac{1}{2}x^{-1/2}$. We replace dx in the integral, but observe that replacing $x^{-1/2}$ works equally well.

We see that $dx = 2x^{1/2}\,du$ and write

$$\int \frac{(3+\sqrt{x})^3}{\sqrt{x}}\,dx = \int \frac{(3+x^{1/2})^3}{x^{1/2}}\,dx$$

$$= \int \frac{u^3}{x^{1/2}} \, (2x^{1/2} \, du)$$
$$= \int 2u^3 \, du$$
$$= \frac{2}{3+1} u^{3+1} + C$$
$$= \frac{2}{4} u^4 + C$$
$$= \frac{1}{2} u^4 + C$$
$$= \boxed{\frac{1}{2}(3+\sqrt{x})^4 + C}$$

2. Evaluate $\int 6e^{\tan(13x)} \sec^2(13x) \, dx$.

Solution: Take $u = \tan(13x)$, then $\frac{du}{dx} = 13\sec^2(13x)$. Solving for dx, we have

$$dx = \frac{du}{13\sec^2(13x)}.$$

Next, we write

$$\int 6e^{\tan(13x)} \sec^2(13x) \, dx = \int 6e^u \sec^2(13x) \left(\frac{du}{13\sec^2(13x)} \right)$$
$$= \int \frac{6}{13} e^u \, du$$
$$= \frac{6}{13} e^u + C$$
$$= \boxed{\frac{6}{13} e^{\tan(13x)} + C}$$

3. A pork roast is removed from the freezer and left on the counter to defrost. The temperature of the pork roast is $-4°C$ when it was removed from the freezer, and t hours later was increasing at a rate of

$$T'(t) = 10.6e^{-0.3t°} C/\text{hour}.$$

Assume the pork is defrosted when its temperature reaches $11°C$. How long does it take for the pork roast to defrost? (Estimated answer rounded off to 4 decimal places.)

Solution: Our goal here is to find the time t such that $T(t) = 11$ (which is the temperature at which the roast is defrosted). We are given T' and that $T(0) = -4$ (because this is the initial temperature of the roast). We need to integrate T and use the initial condition $T(0) = -4$ to solve for C.

Let $u = -0.3t$, then $\dfrac{du}{dt} = -0.3$. So, $dt = -\dfrac{du}{0.3}$ and

$$\begin{aligned}\int 10.6e^{-0.3t}\,dt &= \int 10.6e^{u}\left(-\frac{du}{0.3}\right)\\ &= \int -\frac{10.6}{0.3}e^{u}\,du\\ &= -\frac{10.6}{0.3}e^{u} + C\\ &= -\frac{10.6}{0.3}e^{-0.3t} + C\end{aligned}$$

Now, since $T(0) = -4$, we see that

$$\begin{aligned}-4 &= -\frac{10.6}{0.3}\underbrace{e^{-0.3(0)}}_{1} + C\\ \Rightarrow\quad -4 + \frac{10.6}{0.3} &= C\end{aligned}$$

Therefore, our temperature function is given by

$$T(t) = -\frac{10.6}{0.3}e^{-0.3t} - 4 + \frac{10.6}{0.3}.$$

Our final step is to find t such that $T(t) = 11$. Write

$$\begin{aligned}11 &= -\frac{10.6}{0.3}e^{-.3t} - 4 + \frac{10.6}{0.3}\\ \Rightarrow\quad 11 + 4 - \frac{10.6}{0.3} &= -\frac{10.6}{0.3}e^{-0.3t}\\ \Rightarrow\quad -\frac{(0.3)\left(15 - \frac{10.6}{0.3}\right)}{10.6} &= e^{-0.3t}\\ \Rightarrow\quad \ln\left(-\frac{(0.3)\left(15 - \frac{10.6}{0.3}\right)}{10.6}\right) &= -0.3t\\ \Rightarrow\quad -\frac{1}{0.3}\ln\left(-\frac{(0.3)\left(15 - \frac{10.6}{0.3}\right)}{10.6}\right) &= t\end{aligned}$$

which implies

$$t \approx \boxed{1.8419 \text{ hours}}$$

Lesson 1B: Integration by Substitution (II)

1. Definite Integration Via u-substitution

We dealt with indefinite integration via u-substitution and now we address definite integration via u-substitution. In the integral

$$\int_0^1 6x(x^2-1)^2\,dx,$$

the dx (called a **differential**) tells us that 0 and 1 are bounds for x, that is, $0 \le x \le 1$. We should take $u = x^2 - 1$, which means that

$$\frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{du}{2x}.$$

But we **cannot** write

$$\int_{\boxed{0}}^{\boxed{1}} 6x(x^2-1)^2\,dx = \int_{\boxed{0}}^{\boxed{1}} 3u^2\,du$$

because this implies $0 \le u \le 1$ — which is not necessarily true. Since $u = x^2 - 1$, $u \ne x$ which means the bounds for u **shouldn't** be the same as the bounds for x. All this is saying is that we must change the bounds if we change the variable.

We can address this change of bounds in a couple of different ways.

Method 1: Rewrite the bounds in terms of u

Again, $\int_0^1 6x(x^2-1)^2\,dx$ has bounds **in terms of** x. Since $u = x^2 - 1$, we can evaluate u at $x = 0$ and $x = 1$. Write

$$u(0) = 0^2 - 1 = -1$$

$$u(1) = 1^2 - 1 = 0$$

Then,

$$\begin{aligned}\int_0^1 6x(x^2-1)^2\,dx &= \int_{u(0)}^{u(1)} 3u^2\,du \\ &= \int_{-1}^{0} 3u^2\,du \\ &= \frac{3}{2+1}u^{2+1}\Big|_{-1}^{0}\end{aligned}$$

$$= \frac{3}{3}u^3\Big|_{-1}^{0}$$

$$= u^3\Big|_{-1}^{0}$$

$$= 0^3 - (-1)^3$$

$$= -(-1)$$

$$= \boxed{1}$$

With this method, you never need to return to the original variable.

Method 2: Return to the original variable. Write

$$\int_0^1 6x(x^2-1)^2\,dx = \int_{u(0)}^{u(1)} 3u^2\,du$$

$$= \frac{3}{2+1}u^{2+1}\Big|_{u(0)}^{u(1)}$$

$$= \frac{3}{3}u^3\Big|_{u(0)}^{u(1)}$$

$$= u^3\Big|_{u(0)}^{u(1)}$$

$$= (x^2-1)^3\Big|_0^1$$

$$= (1^2-1)^3 - (0^2-1)^3$$

$$= (0)^3 - (-1)^3$$

$$= -(-1)$$

$$= \boxed{1}$$

In this method, you don't need to determine the bounds in terms of u.

EXAMPLES.

1. Evaluate $\displaystyle\int_{-1}^{2} 24(x^2-2)(x^3-6x)^4\,dx$.

Solution: Choosing the correct u here might seem a little tricky. The only function that has an inside and an outside function is $(x^3-6x)^4$, so take $u = x^3 - 6x$. Then $\dfrac{du}{dx} = 3x^2 - 6$. We use the method of solving for dx:

$$dx = \frac{du}{3x^2 - 6}.$$

Next, we need to address the bounds of this integral. We go through the two methods.

Method 1: Rewrite the bounds in terms of u

$u = x^3 - 6x$ and $x = -1$, $x = 2$ which means

$$u(-1) = (-1)^3 - 6(-1) = -1 + 6 = 5$$

$$u(2) = (2)^3 - 6(2) = 8 - 12 = -4$$

Now, we see

$$\begin{aligned}
\int_{-1}^{2} 24(x^2-2)(x^3-6x)^4\, dx &= \int_{u(-1)}^{u(2)} 24(x^2-2)u^4 \underbrace{\left(\frac{du}{3x^2-6}\right)}_{dx} \\
&= \int_{5}^{-4} 24\frac{x^2-2}{3(x^2-2)}u^4\, du \\
&= \int_{5}^{-4} 8u^4\, du \\
&= \frac{8}{4+1}u^{4+1}\Big|_{5}^{-4} \\
&= \frac{8}{5}u^5\Big|_{5}^{-4} \\
&= \frac{8}{5}\left[(-4)^5 - (5)^5\right] \\
&= \frac{8}{5}\left[-1024 - 3125\right] \\
&= \frac{8}{5}\left[-4149\right] \\
&= \boxed{-\frac{33{,}192}{5}}
\end{aligned}$$

Method 2: Return to the original variable

We have

$$\begin{aligned}
\int_{-1}^{2} 24(x^2-2)(x^3-6x)^4\, dx &= \int_{u(-1)}^{u(2)} 24(x^2-2)u^4 \underbrace{\left(\frac{du}{3x^2-6}\right)}_{dx} \\
&= \int_{u(-1)}^{u(2)} 24\frac{x^2-2}{3(x^2-2)}u^4\, du
\end{aligned}$$

$$
\begin{aligned}
&= \int_{u(-1)}^{u(2)} 8u^4\,du \\
&= \frac{8}{4+1}u^{4+1}\Big|_{u(-1)}^{u(2)} \\
&= \frac{8}{5}u^5\Big|_{u(-1)}^{u(2)} \\
&= \frac{8}{5}(x^3-6x)^5\Big|_{-1}^{2} \\
&= \frac{8}{5}(2^3-6(2))^5 - \frac{8}{5}((-1)^3-6(-1))^5 \\
&= \frac{8}{5}(8-12)^5 - \frac{8}{5}(-1+6)^5 \\
&= \frac{8}{5}(-4)^5 - \frac{8}{5}(5)^5 \\
&= \frac{8}{5}(-1024) - \frac{8}{5}(3125) \\
&= -\frac{8192}{5} - \frac{25{,}000}{5} \\
&= \boxed{-\frac{33{,}192}{5}}
\end{aligned}
$$

2. Evaluate $\displaystyle\int_0^4 3e^{2x}\,dx$.

Solution: Take $u = 2x$, then $\dfrac{du}{dx} = 2$ which means $dx = \dfrac{du}{2}$.

Method 1: Rewrite the bounds in terms of u

If $u = 2x$ and $x = 0$, $x = 4$, then

$$u(0) = 2(0) = 0$$

$$u(4) = 2(4) = 8$$

So, we have

$$
\begin{aligned}
\int_0^4 3e^{2x}\,dx &= \int_{u(0)}^{u(4)} 3e^u \underbrace{\left(\frac{du}{2}\right)}_{dx} \\
&= \int_0^8 \frac{3}{2}e^u\,du
\end{aligned}
$$

$$= \frac{3}{2}e^u \Big|_0^8$$

$$= \frac{3}{2}e^8 - \frac{3}{2}\underbrace{e^0}_{1}$$

$$= \boxed{\frac{3}{2}e^8 - \frac{3}{2}}$$

Method 2: Return to the original variable

Write

$$\int_0^4 3e^{2x}\,dx = \int_{u(0)}^{u(4)} 3e^u \underbrace{\left(\frac{du}{2}\right)}_{dx}$$

$$= \int_{u(0)}^{u(4)} \frac{3}{2}e^u\,du$$

$$= \frac{3}{2}e^u \Big|_{u(0)}^{u(4)}$$

$$= \frac{3}{2}e^{2x} \Big|_0^4$$

$$= \frac{3}{2}e^{2(4)} - \frac{3}{2}e^{2(0)}$$

$$= \frac{3}{2}e^8 - \frac{3}{2}\underbrace{e^0}_{1}$$

$$= \boxed{\frac{3}{2}e^8 - \frac{3}{2}}$$

There is also a cute trick with u-sub that is at times necessary when u is a linear function (of the form $u = mx + b$).

Ex 1. Evaluate $\int x(3x+4)^7\,dx$.

If we reasonably choose $u = 3x + 4$, then

$$\int x(3x+4)^7\,dx = \int xu^7 \underbrace{\left(\frac{1}{3}\,du\right)}_{dx}.$$

We have not completely eliminated x, but what other choice of u do we have?

Luckily, we need only observe that if $u = 3x + 4$, then $x = \dfrac{u-4}{3}$. Hence,

$$\begin{aligned}
\int x(3x+4)^7\,dx &= \int \underbrace{\left(\frac{u-4}{3}\right)}_{x} u^7 \left(\frac{1}{3}\,du\right) \\
&= \frac{1}{3\cdot 3}\int (u-4)u^7\,du \\
&= \frac{1}{9}\int (u^8 - 4u^7)\,du \\
&= \frac{1}{9}\left[\frac{1}{8+1}u^{8+1} - \frac{4}{7+1}u^{7+1}\right] + C \\
&= \frac{1}{9}\left[\frac{1}{9}u^9 - \frac{4}{8}u^8\right] + C \\
&= \frac{1}{81}u^9 - \frac{4}{72}u^8 + C \\
&= \frac{1}{81}u^9 - \frac{1}{18}u^8 + C \\
&= \boxed{\frac{1}{81}(3x+4)^9 - \frac{1}{18}(3x+4)^8 + C}
\end{aligned}$$

Examples.

3. Evaluate $\displaystyle\int_0^3 \frac{3x}{\sqrt{x+1}}\,dx$.

Solution: This integral is a u-sub problem with $u = x+1$, $\dfrac{du}{dx} = 2$, and $x = u - 1$. Write

$$\begin{aligned}
\int_0^3 \frac{3x}{\sqrt{x+1}}\,dx &= \int_{u(0)}^{u(3)} \frac{3(u-1)}{u^{1/2}}\,du \\
&= 3\int_{u(0)}^{u(3)} \frac{u-1}{u^{1/2}}\,du \\
&= 3\int_{u(0)}^{u(3)} \left[\frac{u}{u^{1/2}} - \frac{1}{u^{1/2}}\right] du \\
&= 3\int_{u(0)}^{u(3)} \left[u^{1/2} - u^{-1/2}\right] du \\
&= 3\left(\left(\frac{1}{1/2+1}\right)u^{1/2+1} - \left(\frac{1}{-1/2+1}\right)u^{-1/2+1}\right)\Bigg|_{u(0)}^{u(3)} \\
&= 3\left(\left(\frac{1}{3/2}\right)u^{3/2} - \left(\frac{1}{1/2}u^{1/2}\right)\right)\Bigg|_{u(0)}^{u(3)}
\end{aligned}$$

$$= 3\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Big|_{u(0)}^{u(3)}$$

$$= 3\left(\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2}\right)\Big|_{0}^{3}$$

$$= 3\left(\frac{2}{3}(3+1)^{3/2} - 2(3+1)^{1/2}\right) - 3\left(\frac{2}{3}(0+1)^{3/2} - 2(0+1)^{1/2}\right)$$

$$= 3\left(\frac{2}{3}(4)^{3/2} - 2(4)^{1/2}\right) - 3\left(\frac{2}{3}\underbrace{(1)^{3/2}}_{1} - 2\underbrace{(1)^{1/2}}_{1}\right)$$

$$= 2(2)^3 - 6(2) - 2 + 6$$

$$= 2(8) - 12 - 2 + 6$$

$$= 16 - 12 - 2 + 6$$

$$= \boxed{8}$$

4. If the area of the region under the curve

$$y = \frac{1}{5x+2}$$

over the interval $0 \leq x \leq a$ is 2, then what is a?

Solution: The area of the region under this curve over the interval $[0, a]$ is simply the integral

$$\int_0^a \frac{1}{5x+2}\,dx.$$

In particular, we are told that this integral is equal to 10. For this integral, we take $u = 5x + 2$ with $\frac{du}{dx} = 5$ and write

$$\int_0^a \frac{1}{5x+2}\,dx = \int_{u(0)}^{u(a)} \frac{1}{u}\underbrace{\left(\frac{1}{5}\,du\right)}_{dx}$$

$$= \frac{1}{5}\int_{u(0)}^{u(a)} \frac{1}{u}\,du$$

$$= \frac{1}{5}\ln|u|\Big|_{u(0)}^{u(a)}$$

$$= \frac{1}{5}\ln|5x+2|\Big|_{0}^{a}$$

$$= \frac{1}{5}\ln|5a+2| - \frac{1}{5}\ln|5(0)+2|$$

$$= \frac{1}{5}\ln|5a+2| - \frac{1}{5}\ln|2|$$
$$= \frac{1}{5}\left(\ln(5a+2) - \ln 2\right)$$
$$= \frac{1}{5}\ln\left(\frac{5a+2}{2}\right)$$

because we assume that $a > 0$ and so $5a + 2 > 0$.

Putting this together,

$$2 = \int_0^a \frac{1}{5x+2} = \frac{1}{5}\ln\left(\frac{5a+2}{2}\right).$$

Hence, we need only solve for a:

$$2 = \frac{1}{5}\ln\left(\frac{5a+2}{2}\right)$$
$$\Rightarrow \quad 10 = \ln\left(\frac{5a+2}{2}\right)$$
$$\Rightarrow \quad e^{10} = \frac{5a+2}{2}$$
$$\Rightarrow \quad 2e^{10} = 5a+2$$
$$\Rightarrow \quad 2e^{10} - 2 = 5a$$
$$\Rightarrow \quad \boxed{\frac{2e^{10}-2}{5}} = a$$

5. Evaluate $\int 27t^2\sqrt{3t+4}\,dt$.

Solution: Let $u = 3t + 4$, then $du = 3\,dt \Rightarrow \dfrac{du}{3} = dt$. But we're not quite done yet since we **cannot** write

$$\int 27t^2\sqrt{3t+4}\,dt = \int 27t^2\sqrt{u}\left(\frac{du}{3}\right)$$

because we must *completely* eliminate the original variable. Instead, we observe that

$$u = 3t + 4 \Rightarrow t = \frac{u-4}{3}$$

and write

$$\begin{aligned}\int 27t^2\sqrt{3t+4}\,dt &= \int 27\left(\frac{u-4}{3}\right)^2\sqrt{u}\left(\frac{du}{3}\right)\\ &= \int\left(\frac{27}{3^2\cdot 3}\right)(u-4)^2\sqrt{u}\,du\\ &= \int (u-4)^2\sqrt{u}\,du.\end{aligned}$$

So now we can solve the integral as usual, even if it requires more computation than we would like.

$$\begin{aligned}\int (u-4)^2\sqrt{u}\,du &= \int (u^2-8u+16)\sqrt{u}\,du\\ &= \int (u^{4/2}-8u^{2/2}+16)u^{1/2}\,du\\ &= \int (u^{5/2}-8u^{3/2}+16u^{1/2})\,du\\ &= \frac{2}{7}u^{7/2}-8\left(\frac{2}{5}\right)u^{5/2}+16\left(\frac{2}{3}\right)u^{3/2}+C\\ &= \frac{2}{7}u^{7/2}-\frac{16}{5}u^{5/2}+\frac{32}{3}u^{3/2}+C\\ &= \boxed{\frac{2}{7}(3t+4)^{7/2}-\frac{16}{5}(3t+4)^{5/2}+\frac{32}{3}(3t+4)^{3/2}+C}.\end{aligned}$$

You would get full credit on quiz for leaving your answer in the above form.

Lesson 2: Integration by Substitution (III)

1. Solutions to In-Class Examples

EXAMPLE 1. A certain plant grows at a rate

$$H'(t) = \frac{1}{\sqrt[3]{9t+2}} \text{ inches per day,}$$

t days after it was planted. How many inches will the height of the plant change on the third day? Round your answer to the nearest thousandth.

Solution: The key to this problem is determining what t-values correspond to the third day after the plant is planted. Consider

$$\underbrace{\text{Day 1}}_{0\le t<1}, \quad \underbrace{\text{Day 2}}_{1\le t<2}, \quad \underbrace{\text{Day 3}}_{2\le t<3}.$$

Hence, if we want to determine how the height of the plant *changes* on the third day, then we are considering the integral

$$\int_2^3 \frac{1}{\sqrt[3]{9t+2}}\,dt.$$

We rewrite the cube root as a fractional exponent:

$$\frac{1}{\sqrt[3]{9t+2}} = \frac{1}{(9t+2)^{1/3}} = (9t+2)^{-1/3}.$$

Now, we approach this integral via u-sub. Let $u = 9t+2$, then $\dfrac{du}{dt} = 9$ which means

$$dt = \frac{du}{9}.$$

We have bounds in terms of t but, because we are changing variables, we need to remake them into bounds in terms of u:

$$u(2) = 9(2) + 2 = 18 + 2 = 20$$

$$u(3) = 9(3) + 2 = 27 + 2 = 29$$

We write

$$\int_2^3 \frac{1}{\sqrt[3]{9t+2}}\,dt = \int_2^3 (9t+2)^{-1/3}\,dt$$

$$
\begin{aligned}
&= \int_{u(2)}^{u(3)} u^{-1/3} \underbrace{\left(\frac{du}{9}\right)}_{dt} \\
&= \int_{20}^{29} \frac{1}{9} u^{-1/3}\, du \\
&= \frac{1}{9}\left(\frac{1}{-1/3+1}\right) u^{-1/3+1}\Big|_{20}^{29} \\
&= \frac{1}{9}\left(\frac{1}{2/3}\right) u^{2/3}\Big|_{20}^{29} \\
&= \frac{1}{9}\left(\frac{3}{2}\right) u^{2/3}\Big|_{20}^{29} \\
&= \frac{1}{6} u^{2/3}\Big|_{20}^{29} \\
&= \frac{1}{6}(29)^{2/3} - \frac{1}{6}(20)^{2/3} \\
&\approx \boxed{0.35 \text{ inches}}
\end{aligned}
$$

EXAMPLE 2. Suppose that as a yellow car brakes, its velocity is described by

$$v(t) = 2.3e^{1-t} - 0.7 \text{ meters/second.}$$

If the brakes are applied at time $t = 0$ seconds, what is the distance it takes for the car to come to a complete stop. Round your answer to 3 decimal places.

Solution: Observe first that the integral of $v(t)$ is the distance function, denoted $s(t)$. Also note that we have the initial condition $s(0) = 0$ because we assume that, at the time $t = 0$, we have gone no distance. But we aren't quite done yet. Ultimately, we need to compute $s(t_0)$ where t_0 is the time such that $v(t_0) = 0$ (which is when the car will have come to a stop).

We begin by computing $s(t) = \int v(t)\, dt$. To integrate $2.3e^{1-t} - 0.7$, let $u = 1 - t$, then $\frac{du}{dt} = -1$ which implies $du = -dt$. We write

$$
\begin{aligned}
s(t) = \int \left(2.3e^{1-t} - 0.7\right) dt &= \int 2.3e^{1-t}\, dt - \int 0.7\, dt \\
&= \int 2.3e^{u} \underbrace{(-du)}_{dt} - \int 0.7\, dt \\
&= \int -2.3e^{u}\, du - \int 0.7\, dt \\
&= -2.3e^{u} - 0.7t + C \\
&= -2.3e^{1-t} - 0.7t + C
\end{aligned}
$$

Next, since $s(0) = 0$, we have

$$\begin{aligned} 0 &= -2.3e^{1-0} - 0.7(0) + C \\ &= -2.3e + C \\ \Rightarrow \quad 2.3e &= C \end{aligned}$$

Thus, we have

$$s(t) = -2.3e^{1-t} - 0.7t + 2.3e.$$

We now check for what t_0 do we have $v(t_0) = 0$. Write

$$\begin{aligned} 0 &= 2.3e^{1-t_0} - 0.7 \\ \Rightarrow \quad 0.7 &= 2.3e^{1-t_0} \\ \Rightarrow \quad \frac{0.7}{2.3} &= e^{1-t_0} \\ \Rightarrow \quad \ln\left(\frac{0.7}{2.3}\right) &= 1 - t_0 \\ \Rightarrow \quad t_0 &= 1 - \ln\left(\frac{0.7}{2.3}\right) \end{aligned}$$

Finally, we compute $s(t_0)$:

$$\begin{aligned} s\left(1 - \ln\left(\frac{0.7}{2.3}\right)\right) &= -2.3e^{1-(1-\ln(0.7/2.3))} - 0.7\left(1 - \ln\left(\frac{0.7}{2.3}\right)\right) + 2.3e \\ &\approx \boxed{4.019} \end{aligned}$$

DEFINITION 6 (Average Value of a Function over an Interval). If $f(x)$ is defined on an interval $[a, b]$, then the average value of $f(x)$ over $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXAMPLE 3. Find the average value of the function $f(x) = 4.4xe^{x^2}$ over the interval $0 < x < 1.8$. Round your answer to the nearest hundredth.

Solution: By our formula above, the average value of $4.4xe^{x^2}$ over $0 < x < 1.8$ is given by

$$\frac{1}{1.8-0}\int_0^{1.8} 4.4xe^{x^2}\,dx.$$

Let $u = x^2$, then $\frac{du}{dx} = 2x$. Since we have bounds in terms of x, we convert them into bounds in terms of u:

$$u(0) = (0)^2 = 0$$

$$u(1.8) = (1.8)^2 = 3.24$$

Then, we get

$$\begin{aligned}\frac{1}{1.8}\int_0^{1.8} 4.4xe^{x^2}\,dx &= \frac{1}{1.8}\int_{u(0)}^{u(1.8)} 4.4xe^u \underbrace{\left(\frac{du}{2x}\right)}_{dx} \\ &= \frac{1}{1.8}\int_0^{3.24} \frac{4.4}{2}e^u\,du \\ &= \frac{1}{1.8}\int_0^{3.24} 2.2e^u\,du \\ &= \frac{2.2}{1.8}e^u\Big|_0^{3.24} \\ &= \frac{2.2}{1.8}e^{3.24} - \frac{2.2}{1.8}\underset{\substack{\uparrow\\1}}{e^0} \\ &= \frac{2.2}{1.8}e^{3.24} - \frac{2.2}{1.8} \\ &\approx \boxed{29.99}\end{aligned}$$

Example 4. A science geek brews tea at $195°F$, and observes that the temperature $T(t)$ of the tea after t minutes is changing at the rate of

$$T'(t) = -3.5e^{-.04t}°F/\text{min}.$$

What is the average temperature of the tea during the first 16 minutes after being brewed? Round your answer to the nearest hundredth of a degree.

Solution: We want to find the average temperature from $t = 0$ to $t = 16$. We will need to integrate T' and then use the initial condition, $T(0) = 195$ to solve for T. **Then**, we find the average temperature of the tea. We were not asked to find the average *change in temperature*.

First, we find T given that $T(0) = 195$. Let $u = -.04t$, $\frac{du}{dt} = -.04$ and write

$$\begin{aligned}T(t) = \int -3.5e^{-.04t}\,dt &= \int -3.5e^u \underbrace{\left(-\frac{du}{.04}\right)}_{dt} \\ &= \int 87.5e^u\,du \\ &= 87.5e^u + C \\ &= 87.5e^{-.04t} + C\end{aligned}$$

Moreover,

$$\begin{aligned}195 &= 87.5\underbrace{e^{-.04(0)}}_{1} + C \\ &= 87.5 + C\end{aligned}$$

$$\Rightarrow \quad 195 - 87.5 = C$$
$$\Rightarrow \quad 107.5 = C$$

We conclude that

$$T(t) = 87.5e^{-.04t} + 107.5.$$

Second, we find the average value of T from $t = 0$ to $t = 16$. By our formula,

$$\begin{aligned}
&\frac{1}{16-0}\int_0^{16} \left(87.5e^{-.04t} + 107.5\right)\, dt \\
&= \frac{1}{16}\int_0^{16} 87.5e^{-.04t}\, dt + \frac{1}{16}\int_0^{16} 107.5\, dt \\
&= \frac{87.5}{16}\int_0^{16} e^{-.04t}\, dt + \frac{1}{16}\int_0^{16} 107.5\, dt \\
&= \frac{87.5}{16}\int_{u(0)}^{u(16)} e^u \left(-\frac{du}{.04}\right) + \frac{1}{16}\int_0^{16} 107.5\, dt \\
&= -\frac{87.5}{.64}\int_{u(0)}^{u(16)} e^u\, du + \frac{1}{16}\int_0^{16} 107.5\, dt \\
&= -\frac{87.5}{.64}e^u\Big|_{u(0)}^{u(16)} + \frac{107.5}{16}t\Big|_0^{16} \\
&= -\frac{87.5}{.64}e^{-.04t}\Big|_0^{16} + \frac{107.5}{16}t\Big|_0^{16} \\
&= -\frac{87.5}{.64}e^{-.04t} + \frac{107.5}{16}t\Big|_0^{16} \\
&= -\frac{87.5}{.64}e^{-.04(16)} + \left(\frac{107.5}{16}\right)(16) - \left[-\frac{87.5}{.64}e^{-.04(0)} + \left(\frac{107.5}{16}\right)(0)\right] \\
&= -\frac{87.5}{.64}e^{-.04(16)} + 107.5 + \frac{87.5}{.64} \\
&\approx \boxed{172.13^\circ}
\end{aligned}$$

2. Additional Examples

EXAMPLES.

1. Evaluate $\displaystyle\int_{\sqrt{\ln 2}}^{\sqrt{\ln 11}} 2xe^{x^2}\, dx$.

Solution: Let $u = x^2$, then $\frac{du}{dx} = 2x \Rightarrow \frac{du}{2x} = dx$. Further, evaluating u at the bounds for x,

$$u(\sqrt{\ln 2}) = (\sqrt{\ln 2})^2 = \ln 2$$
$$u(\sqrt{\ln 11}) = (\sqrt{\ln 11})^2 = \ln 11$$

So,

$$\begin{aligned} \int_{\sqrt{\ln 2}}^{\sqrt{\ln 11}} 2xe^{x^2}\,dx &= \int_{u(\sqrt{\ln 2})}^{u(\sqrt{\ln 11})} 2xe^u \underbrace{\left(\frac{du}{2x}\right)}_{dx} \\ &= \int_{\ln 2}^{\ln 11} e^u\,du \\ &= e^u\Big|_{\ln 2}^{\ln 11} \\ &= e^{\ln 11} - e^{\ln 2} \\ &= 11 - 2 = \boxed{9}. \end{aligned}$$

2. Find the average value of $f(x) = 2x + 1$ over the interval $0 \le x \le 2$.

Solution: Here, $a = 0$ and $b = 2$, so by our formula,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 (2x+1)\,dx.$$

Thus, the average value of f over $[0, 2]$ is

$$\begin{aligned} \frac{1}{2}\int_0^2 (2x+1)\,dx &= \frac{1}{2}(x^2+x)\Big|_0^2 \\ &= \frac{1}{2}((2)^2+2) - \frac{1}{2}(0^2+0) \\ &= \frac{1}{2}(4+2) \\ &= \boxed{3} \end{aligned}$$

3. Suppose during a hot dog eating contest you eat $3\sqrt{2t+1}$ hot dogs/minute. What is the average number of hot dogs you eat per minute in the first 4 minutes of the contest?

Solution: We are asked to find the average value of the function $3\sqrt{2t+1}$ over the interval $0 \le t \le 4$. By our formula, the average value is given by

$$\frac{1}{4-0}\int_0^4 3\sqrt{2t+1}\,dt.$$

Observe this is a u-sub problem with bounds. Take $u = 2t + 1$, then $\dfrac{du}{dt} = 2 \;\Rightarrow\; \dfrac{du}{2} = dt$. Moreover,

$$u(0) = 2(0) + 1 = 1$$
$$u(4) = 2(4) + 1 = 9$$

Hence,

$$\begin{aligned}
\frac{1}{4-0}\int_0^4 3\sqrt{2t+1}\,dt &= \frac{1}{4}\int_{u(0)}^{u(4)} \frac{3}{2}\sqrt{u}\,du \\
&= \frac{1}{4}\int_1^9 \frac{3}{2}\sqrt{u}\,du \\
&= \frac{3}{8}\left(\frac{2}{3}\right) u^{3/2}\Big|_1^9 \\
&= \frac{1}{4}(9)^{3/2} - \frac{1}{4}(1)^{3/2} \\
&= \frac{27}{4} - \frac{1}{4} \\
&= \frac{26}{4} = \boxed{\frac{13}{2} \text{ hotdogs per minute}}
\end{aligned}$$

Lesson 3: The Natural Logarithmic Function: Integration

1. Review of Natural Log

Natural log (denoted $\ln x$) is the function inverse of e^x, that is,

$$\ln e^x = x \text{ and } e^{\ln x} = x.$$

Ex 1. $\ln e^2 = 2$ and $e^{\ln(37t+4)} = 37t + 4$.

Caution. $\ln(2e^2) \neq 2$ nor $e^{3\ln(37t+4)} \neq 37t + 4$.

Facts about Natural Log

- $\ln e = 1$, $\ln 1 = 0$ because $e = e^1$, $1 = e^0$
- $\ln x$ is defined **only** for x in $(0, \infty)$
  - $\ln 0$ and $\ln(-7)$ do **not** make sense but $\ln 3$ is valid
  - $\ln x$ may **output** negative values; for example,

$$\ln\left(\frac{1}{2}\right) \approx -.693$$

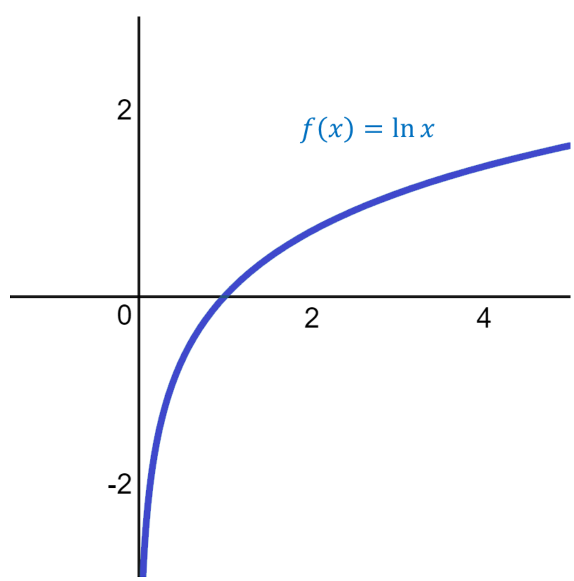


FIGURE 2. Graph of $\ln x$

Properties of $\ln x$

(1) $a \ln b = \ln b^a$

(2) $\ln(ab) = \ln a + \ln b$

(3) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Ex 2.

(1) $3 \ln 2 = \ln 2^3 = \ln 8$

(2) $\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3$

(3) $\ln 3 = \ln\left(\frac{6}{2}\right) = \ln 6 - \ln 2$

Ex 3. The rules above **do not apply** in the case of addition or subtraction, that is,

$$\ln(3+2) \neq \ln 3 + \ln 2 \quad \text{nor} \quad \ln(3-2) \neq \ln 3 - \ln 2.$$

Moreover, $(\ln(3))^6 \neq \ln(3^6)$. You can check these using your calculator.

Recall,

$$\int \frac{1}{x}\, dx = \ln|x| + C \text{ when } x \neq 0.$$

Note that $x^{-1} = \frac{1}{x}$ is the *only* exception to the power rule. Further, notice the absolute value symbols ($|\ \ |$). This is important because $\ln x$ only takes positive inputs and the absolute values allow us to compute integrals like

$$\int_{-6}^{-4} \frac{1}{x}\, dx = \ln|x| \Big|_{-6}^{-4} = \ln|-4| - \ln|-6| = \ln 4 - \ln 6 = \underbrace{\ln\left(\frac{4}{6}\right)}_{\text{by (3)}} = \ln\left(\frac{2}{3}\right)$$

without incident.

We may drop the absolute values if we know the input is always non-negative. That is, $\ln|x^2+1| = \ln(x^2+1)$ because $x^2+1 > 0$ for all x. However, $\ln|3x+1| \neq \ln(3x+1)$ because $3x+1 < 0$ whenever $x < -\frac{1}{3}$.

Ex 4. Observe that

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

But even more is true: for $f(x)$ a function,

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}.$$

More concretely,

$$\frac{d}{dx}(\ln(x^2+2x+1)) = \frac{2x+2}{x^2+2x+1}.$$

REMARK 7. The lesson today is using u-sub with $\ln x$.

2. Examples of u-substitution with Natural Log

EXAMPLES.

1. Evaluate $\int x^{-1}(\ln(x))^6\,dx$.

Solution: Since $x^{-1} = \frac{1}{x}$,

$$\int x^{-1}(\ln(x))^6\,dx = \int \left(\frac{1}{x}\right)(\ln(x))^6\,dx.$$

So, if we take $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x} \;\Rightarrow\; x\,du = dx$. Write

$$\begin{aligned}\int \left(\frac{1}{x}\right)(\ln(x))^6\,dx &= \int \left(\frac{1}{x}\right)(u)^6(x\,du)\\ &= \int u^6\,du\\ &= \frac{1}{7}u^7 + C\\ &= \boxed{\frac{1}{7}(\ln(x))^7 + C}.\end{aligned}$$

2. Evaluate $\int \tan x\,dx$.

Solution: Here, we need to observe that $\tan = \frac{\sin x}{\cos x}$ and so

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

Let $u = \cos x$, then

$$\frac{du}{dx} = -\sin x \;\Rightarrow\; -\frac{du}{\sin x} = dx.$$

Therefore,

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx\\ &= \int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right)\end{aligned}$$

$$\begin{aligned}
&= \int -\frac{1}{u}\,du \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C \\
&= -1 \cdot \ln|\cos x| + C \\
&= \ln|(\cos x)^{-1}| + C \text{ by (1)} \\
&= \boxed{\ln|\sec x| + C}
\end{aligned}$$

since $(\cos x)^{-1} = \dfrac{1}{\cos x} = \sec x$.

3. Evaluate $\displaystyle\int_0^2 \frac{2x}{1+2x^2}\,dx$.

Solution: Let $u = 1 + 2x^2$, then

$$\frac{du}{dx} = 4x \;\Rightarrow\; \frac{du}{4x} = dx.$$

Moreover,

$$\begin{aligned}
u(0) &= 1 + 2(0)^2 = 1 \\
u(2) &= 1 + 2(2)^2 = 1 + 8 = 9
\end{aligned}$$

So we may write

$$\begin{aligned}
\int_0^2 \frac{2x}{1+2x^2}\,dx &= \int_{u(0)}^{u(2)} \frac{2x}{u}\left(\frac{du}{4x}\right) \\
&= \int_1^9 \frac{1}{2u}\,du \\
&= \int_1^9 \frac{1}{2}\left(\frac{1}{u}\right)\,du \\
&= \frac{1}{2}\ln|u|\Big|_1^9 \\
&= \frac{1}{2}\ln(9) - \frac{1}{2}\underbrace{\ln(1)}_{0} \\
&= \ln(9^{1/2}) \\
&= \boxed{\ln 3}
\end{aligned}$$

4. Evaluate $\displaystyle\int_1^{27} \frac{1}{x^{2/3}(1+x^{1/3})}\,dx$.

Solution: We might be tempted here to try to rewrite this as

$$x^{2/3}(1+x^{1/3}) = x^{2/3}\cdot 1 + x^{2/3}x^{1/3} = x^{2/3} + x^{2/3+1/3} = x^{2/3} + x^1$$

and then taking $u = x^{2/3} + x$. But then $\frac{du}{dx} = \left(\frac{2}{3}x^{-1/3} + 1\right)$, which is **not** a factor up to a constant in this integral (try doing this substitution on your own to determine that you can't eliminate the original variable).

Instead, leave the integral as it is and take $u = 1 + x^{1/3}$, then

$$\frac{du}{dx} = \frac{1}{3}x^{-2/3} \;\Rightarrow\; \frac{3\,du}{x^{-2/3}} = dx \;\Rightarrow\; 3x^{2/3}\,du = dx.$$

Thus,

$$\begin{aligned}\int_1^{27} \frac{1}{x^{2/3}(1+x^{1/3})}\,dx &= \int_1^{27} \frac{1}{x^{2/3}u}\left(3x^{2/3}\,du\right)\\ &= \int_{u(1)}^{u(27)} \frac{3}{u}\,du\end{aligned}$$

Since

$$\begin{aligned}u(1) &= 1 + 1^{1/3} = 1 + 1 = 2\\ u(27) &= 1 + 27^{1/3} = 1 + 3 = 4\end{aligned}$$

we have

$$\begin{aligned}\int_1^{27} \frac{1}{x^{2/3}(1+x^{1/3})}\,dx &= \int_{u(1)}^{u(27)} \frac{3}{u}\,du\\ &= \int_2^4 \frac{3}{u}\,du\\ &= 3\ln|u|\Big|_2^4\\ &= 3\ln(4) - 3\ln(2)\\ &= 3(\ln(4) - \ln(2))\\ &= 3\ln\left(\frac{4}{2}\right) \text{ by (3)}\\ &= 3\ln(2)\\ &= \ln(2^3) = \boxed{\ln 8} \text{ by (1)}\end{aligned}$$

5. Find the average value of

$$f(x) = \frac{4(\ln x)^4}{x}$$

over the interval $[1, 6e]$. Round your answer to 3 decimal places.

Solution: Take $u = \ln x$, then $\dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow x\,du = dx$. We do have bounds, $x = 1$ and $x = 6e$, which we change in terms of u:

$$u(1) = \ln(1) = 0$$

$$u(6e) = \ln(6e) = \ln(6) + \ln(e) = \ln(6) + 1$$

Now, since we are asked for the average value, we have

$$\begin{aligned}
\frac{1}{6e-1}\int_1^{6e} \frac{4(\ln x)^4}{x}\,dx &= \frac{1}{6e-1}\int_{u(1)}^{u(6e)} \frac{4u^4}{x}\underbrace{(x\,du)}_{dx} \\
&= \frac{1}{6e-1}\int_0^{\ln(6)+1} 4u^4\,du \\
&= \frac{1}{6e-1}\left(\frac{4}{4+1}\right)u^{4+1}\bigg|_0^{\ln(6)+1} \\
&= \frac{1}{6e-1}\left(\frac{4}{5}\right)u^5\bigg|_0^{\ln(6)+1} \\
&= \frac{4}{5(6e-1)}[(\ln(6)+1)^5 - (0)^5] \\
&= \frac{4(\ln(6)+1)^5}{5(6e-1)} \\
&\approx \boxed{8.862}
\end{aligned}$$

3. Additional Examples

Examples.

1. Evaluate $\displaystyle\int_e^{e^4} \frac{1}{x\ln(x)}\,dx$.

Solution: Take $u = \ln(x)$, then $\dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow x\,du = dx$. Further,

$$u(e) = \ln(e) = \ln(e^1) = 1$$

$$u(e^4) = \ln(e^4) = 4$$

So

$$\begin{aligned}
\int_e^{e^4} \frac{1}{x\ln(x)}\,dx &= \int_{u(e)}^{u(e^4)} \frac{1}{xu}\underbrace{(x\,du)}_{dx} \\
&= \int_{u(e)}^{u(e^4)} \frac{1}{u}\,du
\end{aligned}$$

$$= \int_1^4 \frac{1}{u}\,du$$

$$= \ln|u|\Big|_1^4$$

$$= \ln(4) - \ln(1)$$

$$= \ln(4) - 0 = \boxed{\ln(4)}$$

2. Suppose a factory produces sponges at rate of

$$s'(t) = \frac{3t^2+2}{t^3+2t+1} \text{ thousand sponges/day.}$$

Find the total amount of sponges created in the factory's first week (which is 7 days) of production. Round your answer to the nearest hundred.

Solution: The number of sponges (in thousands) created in the first week is given by

$$\int_0^7 \frac{3t^2+2}{t^3+2t+1}\,dt.$$

Let $u = t^3 + 2t + 1$, then $\frac{du}{dt} = (3t^2+2) \Rightarrow \frac{du}{3t^2+2} = dt$. Write

$$\int_0^7 \frac{3t^2+2}{t^3+2t+1}\,dt = \int_{u(0)}^{u(7)} \frac{3t^2+2}{t^3+2t+1} \underbrace{\left(\frac{du}{3t^2+2}\right)}_{dt}$$

$$= \int_{u(0)}^{u(7)} \frac{1}{u}\,du$$

$$= \ln|u|\Big|_{u(0)}^{u(7)}$$

$$= \ln|t^3+2t+1|\Big|_0^7$$

$$= \ln((7^3) + 2(7) + 1) - \ln((0^3) + 2(0) + 1)$$

$$= \ln((7^3) + 15) - \underset{\substack{\uparrow \\ 0}}{\ln 1}$$

$$= \ln(358) \approx \boxed{5.9 \text{ thousand sponges}}$$

3. Find the area bounded by

$$y = \frac{9\sin x}{3+\cos x}, \quad y = 0, \quad x = \frac{\pi}{7}, \quad x = \frac{6\pi}{7}.$$

Round your answer to 3 decimal places.

Solution: We are asked to integrate

$$\int_{\pi/7}^{6\pi/7} \frac{9\sin x}{3+\cos x}\,dx.$$

Take $u = 3+\cos x$, then $\dfrac{du}{dx} = -\sin x$. Here, we won't change the bounds and instead write

$$\begin{aligned}
\int_{\pi/7}^{6\pi/7} \frac{9\sin x}{3+\cos x}\,dx &= \int_{u(\pi/7)}^{u(6\pi/7)} \frac{9\sin x}{u} \underbrace{\left(-\frac{du}{\sin x}\right)}_{dx} \\
&= \int_{u(\pi/7)}^{u(6\pi/7)} -\frac{9}{u}\,du \\
&= -9\ln|u|\Big|_{u(\pi/7)}^{u(6\pi/7)} \\
&= -9\ln|3+\cos x|\Big|_{\pi/7}^{6\pi/7} \\
&= -9\ln\left|3+\cos\left(\frac{6\pi}{7}\right)\right| - \left[-9\ln\left|3+\cos\left(\frac{\pi}{7}\right)\right|\right] \\
&\approx \boxed{5.578}
\end{aligned}$$

4. Evaluate $\displaystyle\int 11\cot\left(\frac{x}{5}\right)\,dx$.

Solution: We want to observe that

$$\cot\left(\frac{x}{5}\right) = \frac{\cos\left(\frac{x}{5}\right)}{\sin\left(\frac{x}{5}\right)}.$$

We need to determine whether to take

$$u = \cos\left(\frac{x}{5}\right) \qquad \text{or} \qquad u = \sin\left(\frac{x}{5}\right)$$

To choose our u, we must **completely eliminate** the original variable.

Here, we should take $u = \sin\left(\frac{x}{5}\right)$. Then

$$\frac{du}{dx} = \frac{1}{5}\cos\left(\frac{x}{5}\right) \;\Rightarrow\; dx = \frac{5\,du}{\cos\left(\frac{x}{5}\right)}.$$

Next, we write

$$\begin{aligned}\int 11\cot\left(\frac{x}{5}\right)\,dx &= \int 11\frac{\cos\left(\frac{x}{5}\right)}{\sin\left(\frac{x}{5}\right)}\,dx\\ &= \int 11\frac{\cos\left(\frac{x}{5}\right)}{u}\underbrace{\left(\frac{5\,du}{\cos\left(\frac{x}{5}\right)}\right)}_{dx}\\ &= \int\frac{55}{u}\,du\\ &= 55\ln|u|+C\\ &= \boxed{55\ln\left|\sin\left(\frac{x}{5}\right)\right|+C}\end{aligned}$$

5. Suppose a population of penguins changes at a rate of

$$P'(t)=\frac{20e^t}{\ln(2)(1+e^t)}\text{ penguins/year}$$

and that the current population is 2000 penguins.

(a) What is the penguin population after 10 years? Round your answer to the nearest penguin.

Solution: Since $P'(t)$ is the change in population, we need to find $P(t)$ (which is the number of penguins at year t) and then compute $P(10)$.

Write

$$\int P'(t)\,dt=\int\frac{20e^t}{\ln(2)(1+e^t)}\,dt.$$

Let $u=1+e^t$, then $\frac{du}{dt}=e^t \;\Rightarrow\; \frac{du}{e^t}=dt$ which means

$$\begin{aligned}\int\frac{20e^t}{\ln(2)(1+e^t)}\,dt &= \int\frac{20}{\ln(2)}\left(\frac{e^t}{u}\right)\underbrace{\left(\frac{du}{e^t}\right)}_{dt}\\ &= \frac{20}{\ln(2)}\int\frac{1}{u}\,du\\ &= \frac{20}{\ln(2)}\ln|u|+C\\ &= \frac{20}{\ln(2)}\ln|1+e^t|+C.\end{aligned}$$

This lists all the possible $P(t)$, but we need to find the specific $P(t)$ that satisfies $P(0) = 2000$. Write

$$\begin{aligned} & & P(0) &= 2000 \\ \Rightarrow & & \frac{20}{\ln(2)} \ln|1+e^0| + C = P(0) &= 2000 \\ \Rightarrow & & \frac{20}{\ln(2)} \ln(1+1) + C &= 2000 \\ \Rightarrow & & \frac{20\ln(2)}{\ln(2)} + C &= 2000 \\ \Rightarrow & & 20 + C &= 2000 \\ \Rightarrow & & C &= 1980 \end{aligned}$$

Thus,

$$P(t) = \frac{20}{\ln(2)} \ln(1+e^t) + 1980.$$

To answer the question, we compute

$$P(10) = \frac{20}{\ln(2)} \ln(1+e^{10}) + 1980 \approx \boxed{2{,}269 \text{ penguins}}.$$

(b) What is the average change in the penguin population from now to 20 years from now? Round your answer to the nearest penguin.

Solution: We are asked to find the average **change** in the penguin population, which is $P'(t)$. Hence, by our formula for function averages, we need to compute

$$\frac{1}{20-0}\int_0^{20} P'(t)\,dt.$$

By the FTC,

$$\begin{aligned} \frac{1}{20-0}\int_0^{20} P'(t)\,dt &= \frac{1}{20}\left[P(20) - P(0)\right] \\ &= \frac{20}{20\ln(2)}\left[\ln(1+e^{20}) - \ln(1+e^0)\right] \\ &\approx \boxed{28 \text{ penguins}} \end{aligned}$$

Lesson 4: Integration by Parts (I)

1. Integration by Parts

Some functions are the result of differentiation via the product rule. Integrating these functions are tackled by an integration method called **integration by parts**. The key is the following equation:

$$\int u\,dv = uv - \int v\,du \tag{3}$$

MEMORIZE this formula.

Ex 1. Suppose we want to evaluate

$$\int xe^x\,dx.$$

Here, u-substitution fails to produce anything useful. Instead, we need to use integration by parts.

To use integration by parts, we apply equation (3) which means we must identify our u, du, v, dv. The integrand is $u \cdot dv$, which means once u is chosen, **everything else** is dv. Then, we find du by differentiating u and we find v by integrating dv.

$$\begin{aligned} u &= \underline{\hspace{2cm}} & dv &= \text{everything leftover} \\ du &= \text{derivative of } u & v &= \textstyle\int dv \end{aligned}$$

For the example give above, we should take $u = x$. This means that everything leftover in the integral ($e^x\,dx$) is our dv. To get our du and v, we differentiate and integrate respectively:

$$\begin{aligned} u &= x & dv &= e^x\,dx \\ du &= dx & v &= \textstyle\int \underbrace{e^x\,dx}_{dv} = e^x \end{aligned}$$

NOTE 8. In this context, whenever we integrate dv, we always assume the constant is 0.

Now that we have each part of equation (3), we just plug in our functions:

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u}\underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{dx}_{du}.$$

Thus, our integral becomes

$$\underbrace{\int xe^x\,dx}_{\substack{\text{original}\\\text{integral}}} = xe^x - \int e^x\,dx = xe^x - e^x + C$$

NOTE 9. We add a single C at the end of the integration by parts process.

CHECK: $\dfrac{d}{dx}(xe^x - e^x + C) = xe^x$

Integration by parts requires attention to detail and is a difficult method to apply, but it is very useful. The trickiest part of integration by parts is determining the correct choice for u. To choose an appropriate u, we apply the following:

L - Logarithms like $\ln x, \ln(x^3+1)$, etc
I - Inverse trig functions (not for this class)
A - Algebraic functions like x, x^3+x^2+7, polynomials (**NO ROOTS**)
T - Trig functions like $\cos x, \tan x$, etc
E - Exponential functions like $e^x, 2^x$, etc

We use this list as follows:

1. If $\ln x$ appears in the integral, then use $u = \ln x$.

2. If no logarithm appears, then we let u be whatever inverse trig function is present.

3. If neither a logarithm nor an inverse trig function are present, we take u to be an algebraic function.

4. If there is no logarithm, inverse trig, nor algebraic function in the integral, let u be a trig function.

5. If logarithms, inverse trig functions, algebraic functions, and trig functions are all absent, choose u to be an exponential function.

EX 2. To evaluate $\displaystyle\int (3x^2+x-1)e^x\,dx$, we take $u = 3x^2+x-1$ and get the following table:

$$\begin{array}{ll} u = 3x^2+x-1 & dv = e^x\,dx \\ du = (6x+1)\,dx & v = \int e^x\,dx = e^x \end{array}$$

NOTE 10. The above is not really a set of hard and fast rules but rather a rule of thumb for choosing u. For this class, however, LIATE should be sufficient.

EXAMPLES.

1. Evaluate $\displaystyle\int x\ln x\,dx$

Solution: By LIATE, we take $u = \ln x$. Then we get the following table:

$$\begin{aligned} u &= \ln x & dv &= x\,dx \\ du &= \frac{1}{x}\,dx & v &= \textstyle\int x\,dx = \frac{1}{2}x^2 \end{aligned}$$

We apply equation (3),

$$\begin{aligned} \int x \ln x\,dx &= \underbrace{\ln x}_{u} \underbrace{\left(\frac{1}{2}x^2\right)}_{v} - \int \underbrace{\left(\frac{1}{2}x^2\right)}_{v} \underbrace{\left(\frac{1}{x}\,dx\right)}_{du} \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \left(\frac{1}{x}\right)\,dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx \\ &= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C} \end{aligned}$$

2. Find $\displaystyle\int_0^{\pi/3} x \cos x\,dx$

Solution: This is a definite integral but we still apply the same method. By LIATE, $u = x$. So

$$\begin{aligned} u &= x & dv &= \cos x\,dx \\ du &= dx & v &= \textstyle\int \cos x\,dx = \sin x \end{aligned}$$

By equation (3), we write

$$\begin{aligned} \int_0^{\pi/3} x \cos x\,dx &= \underbrace{x}_{u}\underbrace{\sin x}_{v}\Big|_0^{\pi/3} - \int_0^{\pi/3} \underbrace{\sin x}_{v}\underbrace{dx}_{du} \\ &= x \sin x\Big|_0^{\pi/3} - \int_0^{\pi/3} \sin x\,dx \\ &= x \sin x\Big|_0^{\pi/3} - (-\cos x)\Big|_0^{\pi/3} \\ &= x \sin x\Big|_0^{\pi/3} + \cos x\Big|_0^{\pi/3} \\ &= x \sin x + \cos x\Big|_0^{\pi/3} \\ &= \frac{\pi}{3}\underbrace{\sin\left(\frac{\pi}{3}\right)}_{\sqrt{3}/2} + \underbrace{\cos\left(\frac{\pi}{3}\right)}_{1/2} - [\underbrace{0 \sin 0}_{0} + \underbrace{\cos 0}_{1}] \end{aligned}$$

$$= \frac{\pi}{3}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2} - 1 = \boxed{\frac{\sqrt{3}\pi}{6} - \frac{1}{2}}$$

For definite integrals, we have the following integration by parts formula:

$$(4) \qquad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

3. Evaluate $\displaystyle\int_1^e x\ln(x^9)\,dx$

Solution: By LIATE, $u = \ln(x^9)$. Then

$$\frac{d}{dx}\ln(x^9) = \frac{\frac{d}{dx}(x^9)}{x^9} = \frac{9x^8}{x^9} = \frac{9}{x}$$

Thus,

$$\begin{array}{ll} u = \ln(x^9) & dv = x\,dx \\ du = \frac{9}{x}\,dx & v = \int x\,dx = \frac{1}{2}x^2 \end{array}$$

We write

$$\begin{aligned}
\int_1^e x\ln(x^9)\,dx &= \underbrace{\ln(x^9)}_{u}\underbrace{\left(\frac{1}{2}x^2\right)}_{v}\Bigg|_1^e - \int_1^e \underbrace{\left(\frac{1}{2}x^2\right)}_{v}\underbrace{\left(\frac{9}{x}\,dx\right)}_{du} \\
&= \frac{1}{2}x^2\ln(x^9)\Big|_1^e - \frac{9}{2}\int_1^e x^2\left(\frac{1}{x}\right)\,dx \\
&= \frac{1}{2}x^2\ln(x^9)\Big|_1^e - \frac{9}{2}\int_1^e x\,dx \\
&= \frac{1}{2}x^2\ln(x^9)\Big|_1^e - \frac{9}{2}\left(\frac{1}{2}x^2\right)\Big|_1^e \\
&= \frac{1}{2}x^2\ln(x^9) - \frac{9}{4}x^2\Big|_1^e \\
&= \frac{1}{2}e^2\underbrace{\ln(e^9)}_{9} - \frac{9}{4}e^2 - \left[\frac{1}{2}(1)^2\underbrace{\ln(1^9)}_{0} - \frac{9}{4}(1)^2\right] \\
&= \frac{9}{2}e^2 - \frac{9}{4}e^2 + \frac{9}{4} \\
&= \frac{18}{4}e^2 - \frac{9}{4}e^2 + \frac{9}{4} = \boxed{\frac{9}{4}e^2 + \frac{9}{4}}
\end{aligned}$$

4. Find $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

Solution: Sometimes you need to be very clever in how you choose your u even when following LIATE.

If you take $u = x^3$, then $dv = \dfrac{1}{\sqrt{1+x^2}}\,dx$. **But we don't know how to integrate this** dv. Instead, take $u = x^2$ which leaves $dv = \dfrac{x}{\sqrt{1+x^2}}\,dx$. With this choice of dv, we *can* integrate:

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^2}}\,dx &\overset{\substack{u=1+x^2\\du=2x\,dx}}{=} \int \frac{1}{2\sqrt{u}}\,du\\ &= \int \frac{1}{2}u^{-1/2}\,du\\ &= \frac{1}{2}\left(\frac{1}{(-1/2)+1}\right)u^{-1/2+1}\\ &= \frac{1}{2}\left(\frac{1}{1/2}\right)u^{1/2}\\ &= \sqrt{u} = \sqrt{1+x^2}.\end{aligned}$$

So we write

$$\begin{aligned} u &= x^2 & dv &= \frac{x}{\sqrt{1+x^2}}\,dx\\ du &= 2x\,dx & v &= \sqrt{1+x^2}\end{aligned}$$

Now, we can apply equation (3).

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}}\,dx &= \underbrace{(x^2)}_{u}\underbrace{(\sqrt{1+x^2})}_{v} - \int \underbrace{(\sqrt{1+x^2})}_{v}\underbrace{(2x\,dx)}_{du}\\ &= x^2\sqrt{1+x^2} - \underbrace{\int 2x\sqrt{1+x^2}\,dx}_{\diamond}.\end{aligned}$$

Where $(\diamond)$ is, again, a u-substitution problem.

Putting this all together,

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}}\,dx &= x^2\sqrt{1+x^2} - \underbrace{\int 2x\sqrt{1+x^2}\,dx}_{\diamond}\\ &= \boxed{x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2} + C}\end{aligned}$$

$\diamond \displaystyle\int 2x\sqrt{1+x^2}\,dx \overset{\substack{u=1+x^2\\u=2x\,dx}}{=} \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} = \frac{2}{3}(1+x^2)^{3/2}$

Note 11. To use LIATE effectively, choose (1) a u that gets "simpler" when you differentiate and (2) a dv which you know how to integrate.

Remark 12. Observe that we can also integrate $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$ by taking $u = 1+x^2$ then observing $\frac{du}{dx} = 2x$ and $x^2 = u-1$. Next, write

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}}\,dx &= \int \frac{x^2}{\sqrt{1+x^2}}\cdot x\,dx \\ &= \int \frac{u-1}{\sqrt{u}}\cdot x\,\underbrace{\left(\frac{du}{2x}\right)}_{dx} \\ &= \int \frac{u-1}{2\sqrt{u}}\,du\end{aligned}$$

and proceed from the integration from there.

2. u-Substitution vs. Integration by Parts

There are many cases where functions can be integrated by either u-substitution or integration by parts; however, there is usually only one method that will work. Below is a list of different integrals under the method by which they can be integrated. This list is by no means exhaustive but should help compare and contrast when to use what method.

u-Substitution	Integration by Parts
$\int xe^{x^2}\,dx$	$\int xe^{x}\,dx$
$\int \frac{\ln(x)}{x}\,dx$	$\int x\ln x\,dx$
$\int \frac{(\ln(x))^4}{x}\,dx$	$\int \frac{\ln x}{x^4}\,dx$
$\int \sin x(\cos x)^2\,dx$	$\int x\cos x\,dx$

As you go through your homework, add examples to this list for easy review later.

3. Why Integration by Parts Works

Suppose we have functions $f(x)$ and $g(x)$. Then, by the product rule,

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

If we integrate both sides, then

$$\int [f(x)g(x)]'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

But integration undoes differentiation, so

$$\int [f(x)g(x)]'\,dx = f(x)g(x).$$

Therefore,

$$\underbrace{\int [f(x)g(x)]'\,dx}_{f(x)g(x)} = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\Rightarrow \quad f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Now, subtract $\displaystyle\int f'(x)g(x)\,dx$ from both sides,

$$f(x)g(x) - \int f(x)g'(x)\,dx = \int f'(x)g(x)\,dx.$$

Finally, let

$$u = f(x) \qquad dv = g'(x)\,dx$$

$$du = f'(x)\,dx \qquad v = \textstyle\int g'(x)\,dx = g(x)$$

to get

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)\,dx}_{dv} = \underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\,\underbrace{f'(x)\,dx}_{du}.$$

4. Additional Examples

Examples.

1. Evaluate $\displaystyle\int_7^{10} (t-2)e^{3-t}\,dt$.

Solution: By LIATE, take $u = t-2$. Our table is

$$u = t-2 \qquad dv = e^{3-t}\,dt$$

$$du = dt \qquad v = \textstyle\int e^{3-t}\,dt = -e^{3-t}$$

Hence,

$$\int_7^{10} (t-2)e^{3-t}\,dt = \underbrace{(t-2)}_{u}\underbrace{(-e^{3-t})}_{v}\Big|_7^{10} - \int_7^{10} \underbrace{(-e^{3-t})}_{v}\,\underset{du}{\underset{\uparrow}{dt}}$$

$$= -(t-2)e^{3-t}\Big|_7^{10} + \int_7^{10} e^{3-t}\,dt$$

$$= -(t-2)e^{3-t}\Big|_7^{10} - e^{3-t}\Big|_7^{10}$$

$$= -(t-2)e^{3-t} - e^{3-t}\Big|_7^{10}$$

$$= -(10-2)e^{3-10} - e^{3-10} - \left[-(7-2)e^{3-7} - e^{3-7}\right]$$

$$= -8e^{-7} - e^{-7} + 5e^{-4} + e^{-4}$$

$$= \boxed{-9e^{-7} + 6e^{-4}}$$

2. Evaluate $\displaystyle\int_{9.1}^{11.1} x(x-10.1)^5\,dx$.

Solution: Although this is in the integration by parts section, this problem is easier tackled by u-substitution. Let $u = x - 10.1$ where $\dfrac{du}{dx} = 1$, $x = u + 10.1$. We do have bounds, which we change:

$$u(9.1) = 9.1 - 10.1 = -1$$

$$u(11.1) = 11.1 - 10.1 = 1$$

Next, we write

$$\int_{9.1}^{11.1} x(x-10.1)^5\,dx = \int_{u(9.1)}^{u(11.1)} (u+10.1)u^5\,\underset{\underset{dx}{\uparrow}}{du}$$

$$= \int_{-1}^{1} (u^6 + 10.1u^5)\,du$$

$$= \frac{1}{6+1}u^{6+1} + \frac{10.1}{5+1}u^{5+1}\Big|_{-1}^{1}$$

$$= \frac{1}{7}u^7 + \frac{10.1}{6}u^6\Big|_{-1}^{1}$$

$$= \frac{1}{7}(1)^7 + \frac{10.1}{6}(1)^6 - \left[\frac{1}{7}(-1)^7 + \frac{10.1}{6}(-1)^6\right]$$

$$= \frac{1}{7} + \frac{10.1}{6} + \frac{1}{7} - \frac{10.1}{6} = \boxed{\frac{2}{7}}$$

3. Evaluate $\displaystyle\int 20x(\ln(6x))^2\,dx$.

Solution: This is an integration by parts problem. By LIATE, we take $u = (\ln(6x))^2$. By the chain rule, du is given by

$$du = \frac{6}{6x}(2\ln(6x))\,dx = \frac{2\ln(6x)}{x}\,dx.$$

Our table is then

$$u = (\ln(6x))^2 \qquad dv = 20x\,dx$$
$$du = \frac{2\ln(6x)}{x}\,dx \qquad v = \int 20x\,dx = 10x^2$$

We write

$$\int 20x(\ln(6x))^2\,dx = \underbrace{(\ln(6x))^2}_{u}\underbrace{(10x^2)}_{v} - \int \underbrace{10x^2}_{v}\underbrace{\left(\frac{2\ln(6x)}{x}\right)dx}_{du}$$
$$= 10x^2(\ln(6x))^2 - \underbrace{\int 20x\ln(6x)\,dx}_{\diamond\diamond}$$

Observe that $(\diamond\diamond)$ is another integration by parts problem. Let $u_1 = \ln(6x)$, then $du_1 = \frac{6}{6x}\,dx = \frac{1}{x}\,dx$. We make another table:

$$u_1 = \ln(6x) \qquad dv_1 = 20x\,dx$$
$$du_1 = \frac{1}{x}\,dx \qquad v_1 = \int 20x\,dx = 10x^2$$

Hence,

$$(\diamond\diamond) = \int 20x\ln(6x)\,dx$$
$$= \underbrace{\ln(6x)}_{u_1}\underbrace{(10x^2)}_{v_1} - \int \underbrace{10x^2}_{v_1}\underbrace{\left(\frac{1}{x}\right)}_{du_1}$$
$$= 10x^2\ln(6x) - \int 10x\,dx$$
$$= 10x^2\ln(6x) - 5x^2 + C$$

Combining this with our work above, we get

$$\int 20x(\ln(6x))^2\,dx = 10x^2(\ln(6x))^2 - \underbrace{\int 20x\ln(6x)\,dx}_{\diamond\diamond}$$
$$= 10x^2(\ln(6x))^2 - (10x^2\ln(6x) - 5x^2) + C$$
$$= \boxed{10x^2(\ln(6x))^2 - 10x^2\ln(6x) + 5x^2 + C}$$

4. Evaluate $\displaystyle\int_1^{e^4} \frac{14\ln x}{4x^2}\,dx$.

Solution: By LIATE, take $u = \ln x$. Then $dv = \dfrac{14}{4x^2}\,dx = \dfrac{7}{2x^2}$ which means

$$\begin{aligned} v &= \int \frac{7}{2x^2}\,dx \\ &= \int \frac{7}{2}x^{-2}\,dx \\ &= \frac{7}{2}\left(\frac{1}{-2+1}\right)x^{-2+1} \\ &= \frac{7}{2}\left(\frac{1}{-1}\right)x^{-1} \\ &= -\frac{7}{2}x^{-1} \end{aligned}$$

Our table is

$$\begin{aligned} u &= \ln x & dv &= \frac{14}{4x^2}\,dx \\ du = \frac{1}{x} &= x^{-1} & v &= -\frac{7}{2}x^{-1} \end{aligned}$$

Hence,

$$\begin{aligned} \int_1^{e^4} \frac{14\ln x}{4x^2}\,dx &= \underbrace{\ln x}_{u}\underbrace{\left(-\frac{7}{2}x^{-1}\right)}_{v}\Bigg|_1^{e^4} - \int_1^{e^4} \underbrace{\left(-\frac{7}{2}x^{-1}\right)}_{v}\underbrace{\left(x^{-1}\,dx\right)}_{du} \\ &= -\frac{7\ln x}{2x}\Bigg|_1^{e^4} + \int_1^{e^4} \frac{7}{2}x^{-2}\,dx \\ &= -\frac{7\ln x}{2x}\Bigg|_1^{e^4} + \frac{7}{2}\left(\frac{1}{-2+1}\right)x^{-2+1}\Bigg|_1^{e^4} \\ &= -\frac{7\ln x}{2x}\Bigg|_1^{e^4} + \frac{7}{2}\left(\frac{1}{-1}\right)x^{-1}\Bigg|_1^{e^4} \\ &= -\frac{7\ln x}{2x}\Bigg|_1^{e^4} - \frac{7}{2x}\Bigg|_1^{e^4} \\ &= -\frac{7\ln x}{2x} - \frac{7}{2x}\Bigg|_1^{e^4} \\ &= -\frac{7\ln e^4}{2e^4} - \frac{7}{2e^4} - \left[-\frac{7\ln 1}{2(1)} - \frac{7}{2(1)}\right] \\ &= -\frac{7(4)}{2e^4} - \frac{7}{2e^4} + \frac{7}{2} \end{aligned}$$

$$= -\frac{28}{2e^4} - \frac{7}{2e^4} + \frac{7}{2} = \boxed{-\frac{35}{2e^4} + \frac{7}{2}}$$

5. Evaluate $\displaystyle\int \frac{z^2+1}{e^z}\,dz$.

Solution: First, we rewrite the integral a little. Since $\frac{1}{e^z} = e^{-z}$, we can write

$$\int \frac{z^2+1}{e^z}\,dz = \int (z^2+1)e^{-z}\,dz$$
$$= \int (z^2e^{-z} + e^{-z})\,dz$$
$$= \underbrace{\int z^2e^{-z}\,dz}_{(\star)} + \int e^{-z}\,dz.$$

We see that $\int e^{-z}\,dz$ can be done via u-substitution but it is not immediately clear what $\underbrace{\int z^2e^{-z}\,dz}_{(\star)}$ is. We turn our attention to $(\star)$.

Solving $(\star)$: This is an integration by parts problem. By LIATE, we take

$$u = z^2 \qquad dv = e^{-z}\,dz$$
$$du = 2z\,dz \qquad v = -e^{-z}$$

Then we simply plug this into equation (3):

$$(\star) = \int \underbrace{(z^2)}_{u}\underbrace{(e^{-z}\,dz)}_{dv} = \underbrace{(z^2)}_{u}\underbrace{(-e^{-z})}_{v} - \int \underbrace{(-e^{-z})}_{v}\underbrace{(2z\,dz)}_{du}$$
$$= -z^2e^{-z} - (-2)\int ze^{-z}\,dz$$
$$= -z^2e^{-z} + 2\underbrace{\int ze^{-z}\,dz}_{(\star\star)}.$$

Because $\int ze^{-z}\,dz$ does not have an obvious antiderivative and u-substitution fails, we run into another situation wherein we need to use integration by parts. We focus on finding $(\star\star)$.

Solving $(\star\star)$: By LIATE, we take

$$u_1 = z \qquad dv_1 = e^{-z}\,dz$$

$$du_1 = dx \qquad v_1 = -e^{-z}$$

Then by equation (3):

$$\begin{aligned}(\star\star) = \int \underbrace{(z)}_{u_1}\underbrace{(e^{-z})}_{dv_1} &= \underbrace{(z)}_{u_1}\underbrace{(-e^{-z})}_{v_1} - \int \underbrace{(-e^{-z})}_{v_1}\underbrace{(dx)}_{du_1}\\ &= -ze^{-z} - \int -e^{-z}\,dz\\ &= -ze^{-z} - (-1)\int e^{-z}\,dz\\ &= -ze^{-z} + \int e^{-z}\,dz\\ &= -ze^{-z} - e^{-z}.\end{aligned}$$

Finally, we put it all back together.

Putting it all back together: We were originally given the integral $\int \frac{z^2+1}{e^z}\,dz$. By our work, we can now write

$$\begin{aligned}\int \frac{z^2+1}{e^z}\,dz &= \underbrace{\int z^2e^{-z}\,dz}_{(\star)} + \int e^{-z}\,dz\\ &= -z^2e^{-z} + 2\underbrace{\int ze^{-z}\,dz}_{(\star\star)} + \int e^{-z}\,dz\\ &= -z^2e^{-z} + 2(-ze^{-z} - e^{-z}) + \int e^{-z}\,dz\\ &= -z^2e^{-z} - 2ze^{-z} - 2e^{-z} - e^{-z} + C\\ &= \boxed{-z^2e^{-z} - 2ze^{-z} - 3e^{-z} + C}\end{aligned}$$

Lesson 5: Integration by Parts (II)

1. Solutions to In-Class Examples

EXAMPLE 1. Suppose a turtle is moving at a speed of $18(t+1)^3 \ln(t+1)^{1/9}$ miles/hour. How far does the turtle travel in half an hour? Round your answer to the nearest thousandth.

Solution: Here, we want to integrate the speed function $18(t+1)^3 \ln(t+1)^{1/9}$ because we need to know the distance accumulated by the turtle between 0 and $\frac{1}{2}$ hours. This integral is

$$\int_0^{1/2} 18(t+1)^3 \ln(t+1)^{1/9}\, dt.$$

u-substitution will not work here and so we need to use integration by parts. However, we should first simplify our integral. By our rules about $\ln x$,

$$\ln(t+1)^{1/9} = \frac{1}{9}\ln(t+1).$$

We write

$$\int_0^{1/2} 18(t+1)^3 \ln(t+1)^{1/9}\, dt = \int_0^{1/2} \frac{18}{9}(t+1)^3 \ln(t+1)\, dt = \int_0^{1/2} 2(t+1)^3 \ln(t+1)\, dt.$$

Next, by LIATE, we take $u = \ln(t+1)$, which means $dv = 2(t+1)^3\, dt$. Observe that

$$(\star) \qquad \int (t+1)^3\, dt \overset{\substack{w=t+1 \\ dw=dt}}{=} \int w^3\, dt = \frac{1}{4}w^4 = \frac{1}{4}(t+1)^4.$$

Hence, integrating dv,

$$(\star\star) \qquad \int 2(t+1)^3\, dt = 2\int (t+1)^3\, dt = 2\left(\frac{1}{4}\right)(t+1)^4 = \frac{1}{2}(t+1)^4.$$

Thus, our table becomes

$$\begin{array}{ll} u = \ln(t+1) & dv = 2(t+1)^3\, dt \\ du = \dfrac{1}{t+1}\, dt & v \overset{(\star\star)}{=} \dfrac{1}{2}(t+1)^4 \end{array}$$

So we write

$$\int_0^{1/2} 2(t+1)^3 \ln(t+1)\,dt$$

$$= \underbrace{\ln(t+1)}_{u}\underbrace{\left(\frac{1}{2}(t+1)^4\right)}_{v}\Bigg|_0^{1/2} - \int_0^{1/2} \underbrace{\frac{1}{2}(t+1)^4}_{v}\underbrace{\left(\frac{1}{t+1}\right)dt}_{du}$$

$$= \frac{1}{2}(t+1)^4\ln(t+1)\Bigg|_0^{1/2} - \frac{1}{2}\int_0^{1/2}(t+1)^3\,dt$$

$$= \frac{1}{2}(t+1)^4\ln(t+1)\Bigg|_0^{1/2} - \frac{1}{2}\underbrace{\left(\frac{1}{4}(t+1)^4\right)}_{\text{by }(\star)}\Bigg|_0^{1/2}$$

$$= \frac{1}{2}(t+1)^4\ln(t+1) - \frac{1}{8}(t+1)^4\Bigg|_0^{1/2}$$

$$= \frac{1}{2}\left(\frac{1}{2}+1\right)^4\ln\left(\frac{1}{2}+1\right) - \frac{1}{8}\left(\frac{1}{2}+1\right)^4 - \left(\frac{1}{2}(0+1)^4\ln(0+1) - \frac{1}{8}(0+1)^4\right)$$

$$= \frac{1}{2}\left(\frac{3}{2}\right)^4\ln\left(\frac{3}{2}\right) - \frac{1}{8}\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\ln(1) - \frac{1}{8}\right)$$

$$= \frac{3^4}{2^5}\ln\left(\frac{3}{2}\right) - \frac{3^4}{2^7} + \frac{1}{8}$$

$$\approx \boxed{.519 \text{ miles}}$$

Example 2. A factory produces pollution at a rate of $\dfrac{14\ln(7t+1)}{(7t+1)^3}$ tons/week. How much pollution does the factory produce in a day? Round your answer to the nearest hundredth.

Solution: Our function measures output in terms of weeks but we are asked about the pollution produced in a day. Hence, the integral we must compute is

$$\int_0^{1/7} \frac{14\ln(7t+1)}{(7t+1)^3}\,dt.$$

However, this form looks somewhat unwieldy, so we introduce a small cosmetic change to our integral. Set $x = 7t+1$, then $\dfrac{dx}{dt} = 7 \Rightarrow \dfrac{dx}{7} = dt$ with

$$x(0) = 7(0) + 1 = 1$$

$$x(1/7) = 7(1/7) + 1 = 2$$

Thus,

$$\int_0^{1/7} \frac{14\ln(7t+1)}{(7t+1)^3}\,dt = \int_1^2 \frac{14\ln(x)}{x^3}\left(\frac{dx}{7}\right) = \int_1^2 \frac{2\ln(x)}{x^3}\,dx.$$

So now, we need only compute the far right integral and we have solved the problem.

Our integral cannot be computed using u-substitution, so we apply integration by parts. By LIATE, we take $u = \ln(x)$, and so $dv = \frac{2}{x^3}\,dx$. Since

$$v = \int \frac{2}{x^3}\,dt = \int 2x^{-3}\,dt = \frac{2}{-3+1}x^{-3+1} = \frac{2}{-2}x^{-2} = -x^{-2} = -\frac{1}{x^2},$$

our table becomes

$$\begin{array}{ll} u = \ln(x) & dv = \frac{2}{x^3}\,dx \\ du = \frac{1}{x}\,dx & v = -\frac{1}{x^2} \end{array}$$

Therefore, we write

$$\begin{aligned}
\int_1^2 \frac{2\ln(x)}{x^3}\,dx &= \underbrace{\ln(x)}_{u}\underbrace{\left(-\frac{1}{x^2}\right)}_{v}\Bigg|_1^2 - \int_1^2 \underbrace{\left(-\frac{1}{x^2}\right)}_{v}\underbrace{\left(\frac{1}{x}\,dx\right)}_{du} \\
&= -\frac{\ln(x)}{x^2}\Bigg|_1^2 + \int_1^2 \frac{1}{x^3}\,dx \\
&= -\frac{\ln(x)}{x^2}\Bigg|_1^2 - \frac{1}{2x^2}\Bigg|_1^2 \\
&= -\frac{\ln(x)}{x^2} - \frac{1}{2x^2}\Bigg|_1^2 \\
&= -\frac{2\ln(x)+1}{2x^2}\Bigg|_1^2 \\
&= -\frac{2\ln(2)+1}{2(2^2)} - \left(-\frac{2\ln(1)+1}{2(1^2)}\right) \\
&= -\frac{2\ln 2+1}{2(2^2)} + \frac{2\ln(1)+1}{2(1^2)} \\
&= -\frac{1}{8}(2\ln(2)+1) + \frac{1}{2} \\
&= -\frac{1}{4}\ln(2) - \frac{1}{8} + \frac{1}{2} \\
&= -\frac{1}{4}\ln(2) - \frac{1}{8} + \frac{4}{8} \\
&= \frac{3}{8} - \frac{1}{4}\ln(2) \\
&\approx \boxed{.2 \text{ tons}}
\end{aligned}$$

Example 3. Suppose the number of Emerald Ash Borers in Indiana (an invasive species) is increasing at a rate of

$$E(t) = 30t^2 e^t \text{ members/month}$$

where t is the number of months from the start of 2011. ($t = 0$ corresponds to Jan. 1 and, for our purposes, we assume each month is of equal duration.) What is the average population per month of Emerald Ash Borers in Indiana between March and May of 2011? Round your answer to the nearest integer.

Solution: Because we are asked to find the average number of Emerald Ash Borers between March and May of 2011, we will need to set up a definite integral. First, let's determine our bounds. Since $t = 0$ is Jan. 1, consider

$$\underbrace{0 \le t < 1,}_{\text{Jan}} \quad \underbrace{1 \le t < 2,}_{\text{Feb}} \quad \underbrace{2 \le t < 3,}_{\text{March}} \quad \underbrace{3 \le t < 4}_{\text{April}} \quad \underbrace{4 \le t < 5}_{\text{May}}$$

Hence, we ought to integrate from $t = 2$ to $t = 5$. Our formula for average value is then given by

$$\frac{1}{5-2}\int_2^5 30t^2 e^t \, dt.$$

Second, we integrate. Simplifying, our integral is

$$\frac{1}{5-2}\int_2^5 30t^2 e^t \, dx = \frac{1}{3}\int_2^5 30t^2 e^t \, dx = \int_2^5 10t^2 e^t \, dt.$$

This is an integration by parts integral where, by LIATE, we take $u = 10t^2$ and thus $dv = e^t \, dt$. We write

$$u = 10t^2 \qquad dv = e^t \, dt$$

$$du = 20t \qquad v = e^t$$

Continuing, we have

$$\int_2^5 10t^2 e^t \, dt = \underbrace{10t^2}_{u}\underbrace{(e^t)}_{v}\Big|_2^5 - \int_2^5 \underbrace{(e^t)}_{v}\underbrace{(20t \, dt)}_{du}$$

$$= 10t^2 e^t \Big|_2^5 - \underbrace{\int_2^5 20te^t \, dt}_{(*)}$$

which involves *another* integration by parts problem.

We find $(*)$. Let $u_1 = 20t$ and $dv_1 = e^t \, dt$. Our table is then

$$u_1 = 20t \qquad dv_1 = e^t \, dt$$

$$du_1 = 20 \, dt \qquad v_1 = e^t$$

Hence,

$$\int_2^5 20te^t\,dt = \underbrace{20t}_{u_1}\underbrace{e^t}_{v_1}\Big|_2^5 - \int_2^5 \underbrace{e^t}_{v_1}\underbrace{(20\,dt)}_{du_1}$$

$$= 20te^t\Big|_2^5 - \int_2^5 20e^t\,dt$$

$$= 20te^t\Big|_2^5 - 20e^t\Big|_2^5$$

$$= 20te^t - 20e^t\Big|_2^5$$

Returning to our original problem, we have

$$\int_2^5 10t^2e^t\,dt = 10t^2e^t\Big|_2^5 - \underbrace{\int_2^5 20te^t\,dt}_{(*)}$$

$$= 10t^2e^t\Big|_2^5 - \left(20te^t - 20e^t\Big|_2^5\right)$$

$$= 10t^2e^t - 20te^t + 20e^t\Big|_2^5$$

$$= 10(5)^2e^t - 20(5)e^5 + 20e^5 - \left[10(2)^2e^2 - 20(2)e^2 + 20e^2\right]$$

$$= 170e^5 - 20e^2$$

$$\approx \boxed{25,082 \text{ Emerald Ash Borers}}$$

2. Additional Examples

EXAMPLES.

1. Find the area under the curve of $f(x) = x(x-3)^6$ over the interval $0 \leq x \leq 3$.

Solution: The area under a curve is given by a definite integral. In this case, the definite integral is

$$\int_0^3 x(x-3)^6\,dx.$$

This can be computed using integration by parts, but it is easier to use u-substitution.

Take $u = x - 3$, then $du = dx$. Note $u = x - 3 \Rightarrow u + 3 = x$ and so we can write

$$\int_0^3 x(x-3)^6\,dx = \int_{u(0)}^{u(3)} (u+3)u^6\,du.$$

Since $u = x - 3$, evaluating at $x = 0$ and $x = 3$, we find

$$u(0) = 0 - 3 = -3$$

$$u(3) = 3 - 3 = 0$$

Thus,

$$\int_{u(0)}^{u(3)} (u+3)u^6\,du = \int_{-3}^{0} (u+3)u^6\,du.$$

Finally, we evaluate:

$$\begin{aligned}
\int_{-3}^{0} (u+3)u^6\,du &= \int_{-3}^{0} (u^7 + 3u^6)\,du \\
&= \frac{1}{8}u^8 + \frac{3}{7}u^7\Big|_{-3}^{0} \\
&= \frac{1}{8}(0)^8 + \frac{3}{7}(0)^7 - \left(\frac{1}{8}(-3)^8 + \frac{3}{7}(-3)^7\right) \\
&= -\left(\frac{6561}{8} - \frac{6561}{7}\right) \\
&= 6561\left(-\frac{1}{8} + \frac{1}{7}\right) \\
&= 6561\left(-\frac{7}{56} + \frac{8}{56}\right) \\
&= 6561\left(\frac{1}{56}\right) = \boxed{\frac{6561}{56}}
\end{aligned}$$

2. Evaluate $\int 6x^2 \cos(-3x)\,dx$.

Solution: This is another integral wherein we need to apply integration by parts twice.

First, we let $u = 6x^2$, then

$$u = 6x^2 \qquad dv = \cos(-3x)\,dx$$

$$du = 12x\,dx \qquad v = \int \cos(-3x)\,dx = -\frac{1}{3}\sin(-3x)$$

We write

$$\int 6x^2\cos(-3x)\,dx = \underbrace{6x^2}_{u}\underbrace{\left(-\frac{1}{3}\sin(-3x)\right)}_{v} - \int \underbrace{-\frac{1}{3}\sin(-3x)}_{v}\underbrace{(12x\,dx)}_{du}$$

$$= -2x^2\sin(-3x) + \underbrace{\int 4x\sin(-3x)\,dx}_{(**)}$$

We need to determine $(**)$, which is a second integration by parts problem.

Next, we let $u_1 = 4x$, then

$$\begin{aligned} u_1 &= 4x & dv_1 &= \sin(-3x)\,dx \\ du_1 &= 4\,dx & v_1 &= \textstyle\int \sin(-3x)\,dx = \frac{1}{3}\cos(-3x) \end{aligned}$$

Thus,

$$\int 4x\sin(-3x)\,dx = \underbrace{4x}_{u_1}\underbrace{\left(\frac{1}{3}\cos(-3x)\right)}_{v_1} - \int \underbrace{-\frac{1}{3}\cos(-3x)}_{v_1}\underbrace{(4\,dx)}_{du_1}$$

$$= -\frac{4}{3}x\cos(-3x) + \int \frac{4}{3}\cos(-3x)\,dx$$

$$= -\frac{4}{3}x\cos(-3x) + \frac{4}{3}\left(\frac{1}{3}\right)\sin(-3x) + C$$

$$= -\frac{4}{3}x\cos(-3x) + \frac{4}{9}\sin(-3x) + C$$

Putting this together, we conclude

$$\int 6x^2\cos(-3x)\,dx = -2x^2\sin(-3x) + \underbrace{\int 4x\sin(-3x)\,dx}_{(**)}$$

$$= \boxed{-2x^2\sin(-3x) + \frac{4}{3}x\cos(-3x) + \frac{4}{9}\sin(-3x) + C}$$

3. Suppose the probability of a gold necklace having a gold purity of $100x$ percent (so $0 \le x \le 1$) is given by

$$P(x) = \frac{9e^3}{e^3 - 4}xe^{-3x}.$$

Find the probability that a gold necklace has a purity of at least 75%. Round your answer to the nearest percent.

Solution: We want our gold necklace to have a purity of **at least** 75%. Hence,

$$75 \le 100x \le 100 \;\Rightarrow\; .75 \le x \le 1.$$

Thus, the question comes down to computing

$$\int_{.75}^{1} \frac{9e^3}{e^3 - 4}xe^{-3x}\,dx = \frac{9e^3}{e^3 - 4}\int_{.75}^{1} xe^{-3x}\,dx.$$

For the moment, let's focus on the integral and forget about the constant out front. We want to solve

$$\int_{.75}^{1} xe^{-3x}\,dx.$$

This is an integration by parts problem, which means it comes down to choosing the correct u and dv. By LIATE, let $u = x$ and $dv = e^{-3x}\,dx$. Integrating dv, we get

$$(***) \qquad v = \int dv = \int e^{-3x}\,dx \overset{\substack{w=-3x \\ dw=-3\,dx}}{=} \int -\frac{1}{3}e^{w}\,du = -\frac{1}{3}e^{w} = -\frac{1}{3}e^{-3x}.$$

So our table becomes

$$\begin{array}{ll} u = x & dv = e^{-3x}\,dx \\ du = dx & v = -\dfrac{1}{3}e^{-3x} \end{array}$$

We write

$$\begin{aligned}
\int_{.75}^{1} xe^{-3x}\,dx &= \underbrace{x}_{u}\underbrace{\left(-\frac{1}{3}e^{-3x}\right)}_{v}\Bigg|_{.75}^{1} - \int_{.75}^{1}\underbrace{\left(-\frac{1}{3}e^{-3x}\right)}_{v}\underset{\overset{\uparrow}{du}}{dx} \\
&= -\frac{1}{3}xe^{-3x}\Big|_{.75}^{1} + \frac{1}{3}\underbrace{\int_{.75}^{1} e^{-3x}\,dx}_{(***)} \\
&= -\frac{1}{3}xe^{-3x}\Big|_{.75}^{1} - \frac{1}{9}e^{-3x}\Big|_{.75}^{1} \\
&= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\Big|_{.75}^{1} \\
&= -\frac{1}{3}e^{-3} - \frac{1}{9}e^{-3} - \left[-\frac{1}{3}(.75)e^{-3(.75)} - \frac{1}{9}e^{-3(.75)}\right] \\
&= \frac{1}{4}e^{-3(.75)} + \frac{1}{9}e^{-3(.75)} - \frac{1}{3}e^{-3} - \frac{1}{9}e^{-3} \\
&= \left(\frac{1}{4} + \frac{1}{9}\right)e^{-3(.75)} - \left(\frac{1}{3} + \frac{1}{9}\right)e^{-3}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{9e^{3}}{e^{3}-4}\int_{.75}^{1} xe^{-3x}\,dx &= \frac{9e^{3}}{e^{3}-4}\left[\left(\frac{1}{4} + \frac{1}{9}\right)e^{-3(.75)} - \left(\frac{1}{3} + \frac{1}{9}\right)e^{-3}\right] \\
&\approx .18
\end{aligned}$$

So the probability is $\boxed{18\%}$.

4. Suppose a certain plant is growing at a rate of te^t inches per day t days after it is planted. What is the height of the plant at the beginning of the third day (assuming it is planted as a seed on the first day)?

Solution: Because of how we are measuring t, we make the observation

$$\underbrace{0 \le t < 1}_{\text{Day 1}}, \quad \underbrace{1 \le t < 2}_{\text{Day 2}}, \quad \underbrace{2 \le t < 3}_{\text{Day 3}}.$$

Let $H(t)$ be the height of the plant t days after it is planted. We know that $H(0) = 0$ (since it was planted as a seed) and we want to find $H(2)$ because $t = 2$ corresponds to the beginning of day 3. Note that $H'(t) = te^t$ because this is the rate of change of the height of the plant. Now, the integral

$$\int te^t \, dt$$

is an integration by parts problem. By LIATE, we choose

$$u = t \qquad dv = e^t \, dt$$

$$du = dt \qquad v = \int e^t \, dt = e^t.$$

So by equation (3),

$$\int te^t \, dt = \underset{\substack{\uparrow \\ u}}{t} \, \underset{\substack{\uparrow \\ v}}{e^t} - \int \underset{\substack{\uparrow \\ v}}{e^t} \, \underset{\substack{\uparrow \\ du}}{dt} = te^t - e^t + C = (t-1)e^t + C.$$

We need to find $H(t)$ given our initial condition $H(0) = 0$.

$$0 = (0-1)e^0 + C = -1 + C \;\Rightarrow\; C = 1.$$

Thus,

$$H(t) = (t-1)e^t + 1.$$

Finally,

$$H(2) = (2-1)e^2 + 1 = e^2 + 1.$$

The plant is $\boxed{e^2 + 1 \text{ inches}}$ tall at the beginning of the third day.

Lesson 6: Diff. Eqns.: Solns, Growth and Decay, & Sep. of Variables

1. Separable Differential Equations

DEFINITION 13. A **differential equation** is an equation that includes one or more derivatives of a function.

Ex 1. $\frac{dy}{dt} = 8y$, $y' = t\cos y$, $y' = x^3y + xy^2$, and $\frac{dy}{dt} = (\cos t)y + t^2 + \frac{1}{3}y$ are all examples of differential equations.

DEFINITION 14. A differential equation is called **separable** if it can be written in the form $\frac{dy}{dx} = f(x)g(y)$.

Ex 2.

- $\frac{dy}{dt} = 8y$, $y' = t\cos y$ are separable
- $y' = x^3y + xy^2$, $\frac{dy}{dt} = (\cos t)y + t^2 + \frac{1}{3}y$ are **NOT** separable

We may think of separability as being able to move one variable to a one side of the equal sign and the other variable to the other, thinking of the equal sign as separating the variables.

Ex 3. We show that $\frac{dy}{dx} = x^2e^{3y-2x^4}$ is separable.

We need to rewrite this as $\frac{dy}{dx} =$ (function of x) $\times$ (function of y). Write

$$\begin{aligned} x^2e^{3y-2x^4} &= x^2e^{3y}e^{-2x^4} \\ &= \frac{x^2e^{3y}}{e^{2x^4}} \\ &= \underbrace{\left(\frac{x^2}{e^{2x^4}}\right)}_{\substack{\uparrow \\ \text{only} \\ x}} \underbrace{\left(e^{3y}\right)}_{\substack{\uparrow \\ \text{only} \\ y}} \end{aligned}$$

So this differential equation is separable.

DEFINITION 15. A **solution** to a differential equation is a function that you can plug into the differential equation and make the equal sign be true.

Ex 4. A solution to

$$\frac{dy}{dx} = \frac{x}{y}$$

is $y(x) = \sqrt{x^2 - 17}$ because

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}(y(x)) &= \frac{d}{dx}\underbrace{\left(\sqrt{x^2-17}\right)}_{y(x)} \\
&= \frac{d}{dx}(x^2-17)^{1/2} \\
&= \frac{1}{2}(x^2-17)^{-1/2}(2x) \text{ by chain rule} \\
&= \left(\frac{1}{2}\frac{1}{\sqrt{x^2-17}}\right)(2x) \\
&= \frac{x}{\sqrt{x^2-17}} \\
&= \frac{x}{y} \text{ since } y = \sqrt{x^2-17}
\end{aligned}$$

Definition 16. A **particular solution** is a solution without any unknowns.

Think of this as solving for C after an indefinite integral when you have initial conditions.

Examples.

1. Find the particular solution to the following:

$$\frac{dy}{dx} = \frac{x^2}{y^2}; \text{ if } y = 1 \text{ when } x = 0.$$

Solution: This is a separable differential equation because

$$\frac{x^2}{y^2} = \underset{\substack{\uparrow \\ \text{only} \\ x}}{x^2}\underset{\substack{\uparrow \\ \text{only} \\ y}}{\left(\frac{1}{y^2}\right)}.$$

We like separable differential equations because we can do the following:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2}{y^2} \\
\Rightarrow \quad y^2\frac{dy}{dx} &= x^2 \\
\Rightarrow \quad y^2\,dy &= x^2\,dx \\
\Rightarrow \quad \int y^2\,dy &= \int x^2\,dx
\end{aligned}$$

$$\Rightarrow \quad \frac{1}{3}y^3 = \frac{1}{3}x^2 + C$$

NOTE 17. We only add one C, and it doesn't matter what side you put it on. I tend to put it on the RHS but there is nothing wrong with putting it on the left hand side (LHS).

Now, the *solution* we are looking for is a function $y(x)$. So we need to solve for y. Write

$$\frac{1}{3}y^3 = \frac{1}{3}x^2 + C$$
$$\Rightarrow \quad y^3 = x^3 + 3C$$
$$\Rightarrow \quad y = \sqrt[3]{x^3 + 3C}$$
$$\Rightarrow \quad y \stackrel{(\star)}{=} \sqrt[3]{x^3 + C}$$

We call $y(x) = \sqrt[3]{x^3 + C}$ the **general solution** because there are unknowns. Our final step is to find the appropriate C. We were told that $y = 1$ when $x = 0$, so

$$1 = \sqrt[3]{(0)^3 + C} = \sqrt[3]{C}.$$

Raising both sides to the third power, we get $C = 1$. Going back to our general solution,

$$y = \sqrt[3]{x^3 + C} = \sqrt[3]{x^3 + 1}.$$

Therefore, the *particular solution* is

$$\boxed{y = \sqrt[3]{x^3 + 1}}.$$

NOTE 18. The 1 does **NOT** go on the outside of the cube root.

2. Proportionality Constants

Differential equations are useful in modeling a variety of situations like populations and radioactive decay. A particular type of differential equation that appears frequently in this class is one that involves proportionality constants. A common theme is a substance or population changing *proportionally* to the function of itself. We address what this means through a series of examples.

EX 5. Suppose a population of deer in Pulaski County, Arkansas changes proportionally to itself. Use a differential equation to describe this phenomenon.

($\star$) C is an arbitrary constant **until** we apply an initial condition, like if $y = 1$ when $x = 0$. Until we introduce such a condition, then we can replace $3C$ by C as being equally arbitrary. If this is uncomfortable, replace $3C$ by C' instead.

Let $y(t)$ be the population of deer in Pulaski County at time t. The change in population is described by the derivative of the population function, which is $y'(t) = \frac{dy}{dt}$. The phrase "changes proportionally to itself" is represented by

$$y' = \frac{dy}{dt} = ky$$

where k is called the **proportionality constant**. This constant depends on the specifics of what is being modeled and is often something which must be found when solving for the particular solution.

Ex 6. Suppose after a February snow storm in Madison, Wisconsin the snow *melts* at a rate of 2.4 times the square of the number of inches of snow on the ground. How would we model this situation?

Let $A(t)$ be the number of inches of snow. Since the snow is *melting* at a rate of 2.4 times the square of $A(t)$, our differential equation is given by

$$A' = \frac{dA}{dt} = -2.4A^2.$$

Observe here that $k = -2.4$ because the snow is melting — meaning $A(t)$ is *decreasing* and thus has a negative derivative.

Ex 7. Assume the cost of ice cream changes inversely proportional to the temperature outside. Write down a differential equation to describe this situation.

Suppose $C(x)$ is the cost of ice cream where x is degrees Fahrenheit. Our differential equation is given by

$$C'(x) = \frac{k}{C(x)}.$$

Ex 8. Suppose in a particular group of 10,000 people, the change in number of people in a Ponzi scheme is jointly proportional to the number of people in the scheme and the number of people not in the scheme. Use a differential equation to describe this situation.

Let $P(t)$ be the number of people in the Ponzi scheme at time t. Then, the number of people not in the scheme is given by $10{,}000 - P(t)$. Joint proportionality means

$$P'(t) = kP(t)(10{,}000 - P(t)).$$

3. Basic Examples

Examples.

2. Find the particular solution to the differential equation

$$y' = ky \text{ given } y(0) = 12, y'(0) = 24.$$

<u>**Solution**</u>: Our first step for solving to y is to find k (which is the proportionality constant). We are told that $y(0) = 12$ and $y'(0) = 24$. Because our differential equation is $y'(t) = ky(t)$, this is true for any t, including $t = 0$:

$$24 = y'(0) = ky(0) = 12k.$$

Hence, we have $24 = 12k \;\Rightarrow\; k = 2$. Substituting,

$$y' = 2y.$$

This is a separable differential equation and we can apply the same technique as in Example 1. Write

$$\begin{aligned} y' = \frac{dy}{dt} &= 2y \\ \Rightarrow \quad \frac{1}{y}\frac{dy}{dt} &= 2 \\ \Rightarrow \quad \frac{1}{y}\,dy &= 2\,dt \\ \Rightarrow \quad \int \frac{1}{y}\,dy &= \int 2\,dt \\ \Rightarrow \quad \ln|y| &= 2t + C \end{aligned}$$

There are two equally valid ways to proceed from here: we can first solve for y and then solve for C, or we can first solve for C and then solve for y. In this situation, we will first solve for C and then solve for y. The initial condition $y(0) = 12$ implies that

$$\begin{aligned} \ln|y(t)| &= 2t + C \\ \Rightarrow \quad \ln|y(0)| &= 2(0) + C \text{ if } t = 0 \\ \Rightarrow \quad \ln(\underset{y(0)}{\underset{\uparrow}{12}}) &= C \end{aligned}$$

Next, we need only solve for y.

To undo natural log, we apply e to both sides. So, we write

$$\begin{aligned} \ln|y| &= 2t + \underbrace{\ln(12)}_{C} \\ \Rightarrow \quad e^{\ln|y|} &= e^{2t+\ln(12)} \\ \Rightarrow \quad |y| &= e^{2t+\ln(12)} \\ &= e^{2t}e^{\ln(12)} \\ &= e^{2t}(12) \\ &= 12e^{2t}. \end{aligned}$$

So we have $|y| = 12e^{2t}$.

The absolute value here leaves us with a choice: either

$$y = 12e^{2t} \text{ or } y = -12e^{2t}.$$

But, because $y(0) = 12$, we conclude that

$$\boxed{y = 12e^{2t}}.$$

3. Suppose $P(t)$ is the mass of a radioactive substance at time t. If $P'(t) = -\frac{3}{2}P(t)$, find the half-life of the substance.

DEFINITION 19. The **half-life** of a substance is the amount of *time* it takes for half of the substance to disappear.

Solution: This may appear to be an impossible problem because we are not told how much of the substance we initially have. For the time being, let A be the original amount of the substance. By definition of half-life, we want to find the t such that $P(t) = \frac{A}{2}$. We are given $P'(t) = -\frac{3}{2}P(t)$, which is separable. So we write

$$\begin{aligned} \frac{dP}{dt} &= -\frac{3}{2}P \\ \Rightarrow \quad \frac{1}{P}\frac{dP}{dt} &= -\frac{3}{2} \\ \Rightarrow \quad \frac{1}{P}\,dP &= -\frac{3}{2}\,dt \\ \Rightarrow \quad \int \frac{1}{P}\,dP &= \int \left(-\frac{3}{2}\right)\,dt \\ \Rightarrow \quad \ln|P| &= -\frac{3}{2}t + C \\ \Rightarrow \quad e^{\ln|P|} &= e^{-3t/2+C} \\ \Rightarrow \quad |P| &= e^{-3t/2+C} \end{aligned}$$

Because we never expect a substance to have a negative amount, we can assume that P is never negative and write

$$P = e^{-3t/2+C}.$$

Here, we need to make the following observation: we assumed that at time $t = 0$, we have an amount A. This means $P(0) = A$. But, by our equation above,

$$P(0) = e^{-3(0)/2+C} = \underset{\substack{\uparrow \\ 1}}{e^0}\, e^C = e^C.$$

Therefore, $A = e^C$. Why is this useful? This lets us write

$$\begin{aligned} e^{-3t/2+C} &= \frac{A}{2} \\ \Rightarrow \quad e^{-3t/2}\underset{\substack{\uparrow \\ A}}{e^C} &= \frac{A}{2} \\ \Rightarrow \quad e^{-3t/2}A &= \frac{A}{2} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad e^{-3t/2} &= \frac{1}{2} \\ \Rightarrow \quad \ln e^{-3t/2} &= \ln\left(\frac{1}{2}\right) \\ \Rightarrow \quad -\frac{3}{2}t &= \ln\left(\frac{1}{2}\right) \\ \Rightarrow \quad t &= -\frac{2}{3}\ln\left(\frac{1}{2}\right) \end{aligned}$$

Thus, the half-life of this substance is

$$\boxed{t = -\frac{2}{3}\ln\left(\frac{1}{2}\right)}.$$

Here, we note that the half-life is *independent* of the initial amount of substance. Further, this t might appear to be a negative number (which wouldn't make sense) but note that $\ln x < 0$ whenever $x < 1$.

NOTE 20. [Useful Info]

- If $P'(t) = kP(t)$, then half-life is given by $\dfrac{\ln\left(\frac{1}{2}\right)}{k}$.
- The general solution to $P'(t) = kP(t)$ is $P(t) = Ae^{kt}$ where $P(0) = A$.

4. Find the general solution to

$$\frac{dy}{dt} = k(50 - y).$$

We assume here that $50 - y > 0$.

Solution: This is a separable differential equation and looks very similar to Example 3. However, we need to be very careful when we integrate. Write

$$\begin{aligned} \frac{dy}{dt} &= k(50 - y) \\ \Rightarrow \quad \frac{1}{50 - y}\frac{dy}{dt} &= k \\ \Rightarrow \quad \frac{1}{50 - y}\,dy &= k\,dt \\ \Rightarrow \quad \int \frac{1}{50 - y}\,dy &= \int k\,dt \end{aligned}$$

Observe that $\displaystyle\int \frac{1}{50-y}\,dy$ is a u-substitution problem: let $u = 50 - y$, then $du = -\,dy$ (note the negative). Hence

$$\int \frac{1}{50-y}\,dy = \int -\frac{1}{u}\,du = -\ln|u| = -\ln|50-y|.$$

So

$$\begin{aligned} -\ln|50-y| &= kt + C \\ \Rightarrow \quad \ln|50-y| &= -kt \underbrace{-C}_{(\star\star)} \end{aligned}$$

We now want to solve for y. Write

$$\begin{aligned} \ln|50-y| &\overset{(\star\star)}{=} -kt + C \\ \Rightarrow \quad e^{\ln|50-y|} &= e^{-kt+C} \\ \Rightarrow \quad |50-y| &= e^{-kt+C}. \end{aligned}$$

Since we assumed that $50 - y > 0$, we may drop the absolute values and so our equation becomes

$$50 - y = e^{-kt+C}.$$

Then, solving for y, we have

$$50 = e^{-kt+C} + y \;\Rightarrow\; 50 - e^{-kt+C} = y.$$

If we let $C = -e^{C}$ (or $C'' = -e^{C'}$), our general solution is

$$\boxed{y = 50 + Ce^{-kt}}.$$

4. Additional Examples

Examples.

1. Suppose a pot roast was 175°F when removed from an oven and set in a 70°F room. If after 10 minutes the pot roast is 160°F, what is its temperature after an hour? Round your answer to the 4th decimal place.

Solution: This problem requires Newton's Cooling Formula:

$$\frac{dT}{dt} = k(T - S)$$

where $T(t)$ is the temperature function and S is the ambient (surrounding) temperature.

We can choose to measure time t however we wish. Here, given this particular question, we measure t in minutes. We are given $T(0) = 175$,

($\star\star$) We replace $-C$ by C or by C', whichever is more comfortable.

$T(10) = 160$, $S = 70$. Write

$$\begin{aligned} \frac{dT}{dt} &= k(T-70) \\ \Rightarrow \qquad \frac{1}{T-70}\,dT &= k\,dt \\ \Rightarrow \qquad \int \frac{1}{T-70}\,dT &= \int k\,dt. \end{aligned}$$

We need to compute

$$\int \frac{1}{T-70}\,dT.$$

Let $u = T - 70$, then $du = dT$. So

$$\begin{aligned} \int \frac{1}{T-70}\,dT &= \int \frac{1}{u}\,du \\ &= \ln|u| \\ &= \ln|T-70|. \end{aligned}$$

Thus,

$$\begin{aligned} \int \frac{1}{T-70}\,dT &= \int k\,dt \\ \Rightarrow \qquad \ln|T-70| &= kt + C \end{aligned}$$

We apply e to both sides to undo the natural log. We get

$$\begin{aligned} \underbrace{e^{\ln|T-70|}}_{|T-70|} &= e^{kt+C} = e^{kt}\underbrace{e^{C}}_{C \text{ or } C'} \\ \Rightarrow \qquad T - 70 &= Ce^{kt}. \end{aligned}$$

Here, we drop the absolute value because we assume the temperature of the pot roast doesn't drop below the temperature of the room. Evaluating at $t = 0$, we see that

$$\begin{aligned} \underbrace{T(0)}_{175} - 70 &= \underbrace{e^{k(0)}}_{1} C = C \\ \Rightarrow \qquad 175 - 70 &= C \\ \Rightarrow \qquad 105 &= C. \end{aligned}$$

Hence,

$$T - 70 = 105e^{kt}$$

which becomes

$$T = 105e^{kt} + 70.$$

We're not done yet since to get a *particular solution*, we must find k. Evaluating at $t = 10$, we have

$$\begin{aligned} 160 = T(10) &= 105e^{k(10)} + 70 \\ \Rightarrow \qquad 90 &= 105e^{10k} \\ \Rightarrow \qquad \frac{6}{7} = \frac{90}{105} &= e^{10k} \\ \Rightarrow \qquad \ln\left(\frac{6}{7}\right) &= 10k \\ \Rightarrow \qquad \frac{1}{10}\ln\left(\frac{6}{7}\right) &= k. \end{aligned}$$

Putting this together,

$$T(t) = 105e^{\ln(6/7)t/10} + 70.$$

Now, to find the temperature after an hour, take $t = 60$. Then

$$T(60) = 105e^{\ln(6/7)(60)/10} + 70 \approx \boxed{111.6398^\circ}.$$

2. Solve the initial value problem for y as a function of t when $y' = -t^n$ with $y(0) = 18$ where n is a constant and $n \geq 0$.

Solution: We are told that y is a function of t, which means that $y' = \dfrac{dy}{dt}$. Thus, our differential equation becomes

$$y' = -t^n \;\Rightarrow\; \frac{dy}{dt} = -t^n \;\Rightarrow\; dy = -t^n\,dt.$$

Next, we integrate:

$$y = \int dy = \int -t^n\,dt.$$

Since $n \geq 0$, we know that $n \neq -1$. Hence, by the power rule, we have

$$\int -t^n\,dt = -\frac{1}{n+1}t^{n+1} + C.$$

Thus,

$$y = -\frac{1}{n+1}t^{n+1} + C.$$

Now, because $y(0) = 18$, we have

$$18 = \underbrace{-\frac{1}{n+1}(0)^{n+1}}_{0} + C = C.$$

We conclude that

$$y(t) = \boxed{-\frac{1}{n+1}t^{n+1} + 18}.$$

3. A bacterial culture grows at a rate proportional to its population. Suppose the population is initially 10,000 and after 2 hours the population has grown to 25,000. Find the population of bacteria as a function of time.

Solution: This is asking us to find a particular solution. Let $P(t)$ be the population of bacteria at time t hours. We are told that population grows at a rate proportional to its population, which means we have the following differential equation

$$P'(t) = kP(t)$$

for k the proportionality constant (which we will need to find). Further, we are told

$$P(0) = 10{,}000 \quad \text{and} \quad P(2) = 25{,}000.$$

We will first find the general solution to $P'(t) = kP(t)$ and then use $P(0)$ and $P(2)$ to find the particular solution. We write

$$\begin{aligned} P'(t) &= kP(t) \\ \Rightarrow \quad \frac{dP}{dt} &= kP \\ \Rightarrow \quad \frac{1}{P}\,dP &= k\,dt \\ \Rightarrow \quad \int \frac{1}{P}\,dP &= \int k\,dt \\ \Rightarrow \quad \ln|P| &= kt + C \\ \Rightarrow \quad e^{\ln|P|} &= e^{kt+C} \\ \Rightarrow \quad |P| &= e^{kt+C} \\ \Rightarrow \quad P &= e^{kt+C} = e^C e^{kt} = Ce^{kt} \end{aligned}$$

where we drop the absolute values because we assume the population is never negative. Thus,

$$P(t) = Ce^{kt}.$$

Next, $P(0) = 10{,}000$ implies

$$C\underbrace{e^{k(0)}}_{1} = 10{,}000 \;\Rightarrow\; C = 10{,}000.$$

Further, since $P(2) = 25{,}000$,

$$\begin{aligned} 25{,}000 &= 10{,}000e^{k(2)} \\ \Rightarrow \quad 2.5 &= e^{2k} \\ \Rightarrow \quad \ln(2.5) &= \ln e^{2k} \end{aligned}$$

$$\Rightarrow \quad \ln(2.5) = 2k$$
$$\Rightarrow \quad \frac{\ln(2.5)}{2} = k$$

Therefore,

$$\boxed{P(t) = 10,000e^{\ln(2.5)t/2}}.$$

NOTE 21. We could have also remembered that the general solution to $P'(t) = kP(t)$ is $P(t) = Ce^{kt}$ where $C = P(0)$. Then we could have skipped to the step that used $P(2)$ to find k.

4. Find y such that $y' = -7\ln t$.

Solution: We are asked to find the general solution to $y' = -7\ln t$. Write

$$y' = -7\ln t$$
$$\Rightarrow \quad \frac{dy}{dt} = -7\ln t$$
$$\Rightarrow \quad dy = -7\ln t\,dt$$
$$\Rightarrow \quad \int dy = \int -7\ln t\,dt$$
$$\Rightarrow \quad y = \int -7\ln t\,dt$$

To find y, we need to evaluate

$$\int -7\ln t\,dt$$

which might seem tricky but is actually not difficult. This is an integration by parts problem. Consider the following table:

$$u = \ln t \qquad dv = -7\,dt$$
$$du = \frac{1}{t}\,dt \qquad v = \int -7\,dt = -7t$$

Hence,

$$\int -7\ln t\,dt = \underbrace{\ln t}_{u}\underbrace{(-7t)}_{v} - \int \underbrace{-7t}_{v}\underbrace{\left(\frac{1}{t}\,dt\right)}_{du}$$
$$= -7t\ln t + \int 7\,dt$$
$$= -7t\ln t + 7t + C$$

Thus, we conclude that

$$y = \boxed{-7t\ln t + 7\ln t + C}.$$

4. A radioactive element decays with a half-life of 6 years. If the mass of the element weighs 7 pounds at $t = 0$, find the amount of the element after 13.9 years. Round your answer to 4 decimal places.

Solution: The differential equation which describes the decay of a radioactive element is

$$y' = ky$$

where $y(t)$ is the amount of the element in pounds after t years. By our formulas from Note (20), the half-life of a radioactive element is given by

$$\text{half-life} = \frac{\ln\left(\frac{1}{2}\right)}{k}.$$

Since we know that the half-life is 6, we see that

$$k = \frac{\ln\left(\frac{1}{2}\right)}{6}.$$

If y is the solution to $y' = ky$, then

$$y = y(0)e^{kt} = y(0)e^{\ln(1/2)t/6}.$$

We know that there are 7 pounds of the element at $t = 0$, which is to say that $y(0) = 7$. Hence,

$$y = 7e^{\ln(1/2)t/6}.$$

Finally, we need only compute $y(13.9)$:

$$y(13.9) = 7e^{\ln(1/2)(13.9)/6} \approx \boxed{1.4051 \text{ pounds}}.$$

5. After 10 minutes in Jean-Luc's room, his tea has cooled to 45° Celsius from 100° Celsius. The room temperature is 25° Celsius. How much longer will it take to cool to 37°? (Round your answer to the nearest hundredth).

Solution: We need to use Newton's Cooling Formula again. The differential equation describing the change in the temperature is given by

$$\frac{dT}{dt} = k(T - S)$$

where T is the temperature, t is minutes, and S is the surrounding temperature. We are told that the tea is initially $100°C$, which is to say $T(0) = 100$, and after 10 minutes it cools to $45°C$, $T(10) = 45$. Our goal is to find the time **past this 10 minutes** at which the tea will cool to $37°C$ (as the question asks how much *longer* it will take to cool).

First, we find the generic solution to the differential equation where we take $S = 25°C$. Write

$$\begin{aligned} \frac{dT}{dt} &= k(T-25) \\ \Rightarrow \quad \frac{dT}{T-25} &= k\,dt \\ \Rightarrow \quad \int \frac{dT}{T-25} &= \int k\,dt \end{aligned}$$

For the LHS, we take $u = T - 25$, then

$$\begin{aligned} \int \frac{dT}{T-25} &= \int \frac{du}{u} \\ &= \ln|u| \\ &= \ln|T-25| \\ &= \ln(T-25) \end{aligned}$$

where we drop the absolute values as we assume the T never drops below $25°C$.

Thus, we see that

$$\begin{aligned} \ln(T-25) &= kt + C \\ \Rightarrow \quad e^{\ln(T-25)} &= e^{kt+C} = \underbrace{e^C}_{C \text{ or } C'} e^{kt} \\ \Rightarrow \quad T - 25 &= Ce^{kt} \\ \Rightarrow \quad T &= Ce^{kt} + 25 \end{aligned}$$

Now that we have the general solution, we use the initial conditions $T(0) = 100$ and $T(10) = 45$ to determine C and k. Since $T(0) = 100$,

$$\begin{aligned} 100 &= Ce^{k(0)} + 25 \\ \Rightarrow \quad 75 &= C\underset{\overset{\uparrow}{1}}{e^0} \\ \Rightarrow \quad 75 &= C \end{aligned}$$

which implies $T = 75e^{kt} + 25$.

Next, since $T(10) = 45$, we have

$$\begin{aligned} 45 &= 75e^{k(10)} + 25 \\ \Rightarrow \quad 20 &= 75e^{10k} \\ \Rightarrow \quad \frac{20}{75} &= e^{10k} \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \ln\left(\frac{4}{15}\right) &= \ln e^{10k} \\
\Rightarrow \quad \ln\left(\frac{4}{15}\right) &= 10k \\
\Rightarrow \quad \frac{1}{10}\ln\left(\frac{4}{15}\right) &= k
\end{aligned}$$

From this, we conclude that

$$T = 75e^{\ln(4/15)t/10} + 25.$$

Finally, we determine the t such that $T(t) = 37$. Write

$$\begin{aligned}
37 &= 75e^{\ln(4/15)t/10} + 25 \\
\Rightarrow \quad 12 &= 75e^{\ln(4/15)t/10} \\
\Rightarrow \quad \frac{12}{75} &= e^{\ln(4/15)t/10} \\
\Rightarrow \quad \ln\left(\frac{4}{25}\right) &= \ln e^{\ln(4/15)t/10} \\
\Rightarrow \quad \ln\left(\frac{4}{25}\right) &= \frac{1}{10}\ln\left(\frac{4}{15}\right)t \\
\Rightarrow \quad \frac{10\ln\left(\frac{4}{25}\right)}{\ln\left(\frac{4}{15}\right)} &= t
\end{aligned}$$

Our final answer is

$$t - 10 = \frac{10\ln\left(\frac{4}{25}\right)}{\ln\left(\frac{4}{15}\right)} - 10 \approx \boxed{3.86 \text{ minutes}}.$$

Lesson 7: Differential Equations: Separation of Variables (I)

1. Solutions to In-Class Examples

Example 1. Find $y(t)$ such that

$$\frac{dy}{dt} - 2t^k y = 0$$

where $y(0) = 1$ and $y(1) = e^{2/7}$.

Solution: We separate variables in our equation:

$$\begin{aligned} \frac{dy}{dt} - 2t^k y &= 0 \\ \Rightarrow \quad \frac{dy}{dt} &= 2t^k y \\ \Rightarrow \quad \frac{dy}{y} &= 2t^k\, dt \end{aligned}$$

Next, we set up our integral.

$$\int \frac{dy}{y} = \int 2t^k\, dt.$$

Now, to integrate the RHS, we need to determine whether $k = -1$ or $k \neq -1$. We go on a bit of an aside to discuss why $k \neq -1$.

Suppose for the moment that $k = -1$. Then, we know that t^{-1} to make sense, t needs to avoid 0. Take $t > 0$ and write

$$\begin{aligned} \int \frac{dy}{y} &= \int 2t^{-1}\, dt \\ \Rightarrow \quad \ln|y| &= 2\ln t + C \\ \Rightarrow \quad y &= e^{2\ln t + C} \\ &= e^C e^{2\ln t} \\ &= Ce^{\ln t^2} \\ y &= Ct^2 \end{aligned}$$

Now, if $y(0) = 1$, then

$$1 = C(0)^2 = 0$$

is a **contradiction**. Since math is all about consistency, we have to discount this possibility. So, we conclude that $k \neq -1$.

Since we know $k \neq -1$, we integrate:

$$\begin{aligned} \int \frac{dy}{y} &= \int 2t^k\,dt \\ \Rightarrow \quad \ln|y| &= \frac{2}{k+1}t^{k+1} + C \\ \Rightarrow \quad |y| &= e^{2t^{k+1}/(k+1)+C} \\ &= e^{2t/(k+1)}e^C \\ \Rightarrow \quad |y| &= Ce^{2t^{k+1}/(k+1)} \end{aligned}$$

We assume that $y > 0$ and so we have

$$y = Ce^{2t^{k+1}/(k+1)}.$$

Since $y(0) = 1$, we have

$$1 = C\underbrace{e^{2(0)^{k+1}/(k+1)}}_{1} = C.$$

Thus, $y = e^{2t^{k+1}/(k+1)}$. Further, $y(1) = e^{2/7}$ which means

$$\begin{aligned} e^{2/7} &= e^{2(1)^{k+1}/(k+1)} \\ &= e^{2/(k+1)} \\ \Rightarrow \quad \ln e^{2/7} &= \ln e^{2/(k+1)} \\ \Rightarrow \quad \frac{2}{7} &= \frac{2}{k+1} \\ \Rightarrow \quad \frac{1}{7} &= \frac{1}{k+1} \\ \Rightarrow \quad k+1 &= 7 \\ \Rightarrow \quad k &= 6 \end{aligned}$$

Therefore,

$$\boxed{y = e^{(2/7)t^7}}.$$

Example 2. A clay mug is 1500°F when it is removed from a kiln and placed in a room with a constant temperature of 70°F. After 2 hours, the mug is 1200°F. What is the temperature of the mug after 5 hours? Round your answer to the nearest degree.

Solution: We use Newton's Cooling Formula:

$$\frac{dT}{dt} = k(T - S)$$

where $T(t)$ is the temperature at time t hours and S is the surrounding temperature.

Let $T(t)$ be the temperature of the mug after t hours. Since the room is 70°F, we have

$$\frac{dT}{dt} = k(T - 70).$$

We separate the variables and integrate:

$$\begin{aligned}
\frac{dT}{dt} &= k(T-70) \\
\Rightarrow \quad \frac{dT}{T-70} &= k\,dt \\
\Rightarrow \quad \int \frac{dT}{T-70} &= \int k\,dt \\
\Rightarrow \quad \ln|T-70| &= kt + C \\
\Rightarrow \quad |T-70| &= e^{kt+C} \\
&= e^{kt}e^{C} \\
\Rightarrow \quad |T-70| &= Ce^{kt}
\end{aligned}$$

We may assume that the temperature of the mug never dips below 70°F and so we don't need the absolute values. We have

$$T - 70 = Ce^{kt} \;\Rightarrow\; T = Ce^{kt} + 70.$$

We know $T(0) = 1500$. Thus, we have

$$\begin{aligned}
1500 &= C\underbrace{e^{k\cdot 0}}_{1}+70 \\
\Rightarrow \quad 1500 &= C + 70 \\
\Rightarrow \quad 1430 &= C
\end{aligned}$$

We conclude $T = 1430e^{kt} + 70$ and since $T(2) = 1200$, we may write

$$\begin{aligned}
1200 &= 1430e^{k\cdot 2} + 70 \\
\Rightarrow \quad 1130 &= 1430e^{2k} \\
\Rightarrow \quad \frac{1130}{1430} &= e^{2k} \\
\Rightarrow \quad \ln\left(\frac{113}{143}\right) &= 2k \\
\Rightarrow \quad \frac{1}{2}\ln\left(\frac{113}{143}\right) &= k.
\end{aligned}$$

This means

$$T = 1430e^{\ln(113/143)t/2} + 70.$$

Finally, we compute $T(5)$:

$$T(5) = 1430e^{5\ln(113/143)/2} + 70 \approx \boxed{864°\text{F}}.$$

Example 3. Suppose the volume of a balloon being inflated satisfies

$$\frac{dV}{dt} = 10\sqrt[5]{V^2}$$

where t is time in seconds after the balloon begins to inflate. If the balloon pops when it reaches a volume of 400 cm^3, after how many seconds will the balloon pop? Round your answer to 3 decimal places.

Solution: We are given

$$\frac{dV}{dt} = 10\sqrt[5]{V^2} = 10(V^2)^{1/5} = 10V^{2/5}.$$

So,

$$\begin{aligned} \frac{dV}{dt} &= 10V^{2/5} \\ \Rightarrow \quad \frac{1}{V^{2/5}}\,dV &= 10\,dt \\ \Rightarrow \quad V^{-2/5}\,dV &= 10\,dt \\ \Rightarrow \quad \int V^{-2/5}\,dV &= \int 10\,dt \\ \Rightarrow \quad \left(\frac{1}{-2/5+1}\right)V^{-2/5+1} &= 10t + C \\ \Rightarrow \quad \left(\frac{1}{3/5}\right)V^{3/5} &= 10t + C \\ \Rightarrow \quad \frac{5}{3}V^{3/5} &= 10t + C. \end{aligned}$$

Since $V(0) = 0$,

$$\frac{5}{3}(0)^{3/5} = 10(0) + C \;\Rightarrow\; C = 0.$$

Our equation is then

$$\frac{5}{3}V = 10t.$$

We could solve for V, but observe that our ultimate goal is to find the t such that $V(t) = 400$. Hence, we need only write

$$\frac{5}{3}(\underbrace{400}_{V(t)})^{3/5} = 10t$$

and solve for t. But this is just

$$t = \frac{1}{6}(400)^{3/5} \approx \boxed{6.069 \text{ seconds}}.$$

EXAMPLE 4. A wet towel hung on a clothesline to dry outside loses moisture at a rate proportional to its moisture content. After 1 hour, the towel has lost 15% of its original moisture content. After how long will the towel have lost 80% of its original moisture content? Round your answer to the nearest hundredth.

Solution: Let $M(t)$ be the percentage of the original moisture content the towel has after t hours. Then

$$M'(t) = kM(t) = 100e^{kt}$$

for k the proportionality constant. We know $M(0) = 100$ because we assume at time t that the towel has not lost any moisture content and $M(1) = 100 - 15 = 85$ because after 1 hour the towel is less 15% of its moisture. Our goal is to find the time t such that $M(t) = 100 - 80 = 20$.

The general solution to a differential equation of the form $M'(t) = kM(t)$ is

$$M(t) = M(0)e^{kt}.$$

So, we need to solve

$$M(t) = 100e^{kt}$$

for k. Since $M(1) = 85$,

$$85 = 100e^{k(1)} = 100e^{k},$$

which implies

$$.85 = e^{k} \Rightarrow \ln(.85) = k.$$

We want to find the t such that

$$20 = M(t) = 100e^{\ln(.85)t}.$$

Write

$$.2 = e^{\ln(.85)t},$$

which, after applying ln to both sides, becomes

$$\ln(.2) = \ln(.85)t \Rightarrow t = \frac{\ln(.2)}{\ln(.85)} \approx \boxed{9.9 \text{ hours}}.$$

2. Additional Examples

EXAMPLES.

1. Find the general solution to

$$\frac{dy}{dt} + 50y = 0.$$

Solution: Since we are finding a *general* solution, we are not asked to solve for all the unknowns. Write

$$\begin{aligned} \frac{dy}{dt} + 50y &= 0 \\ \Rightarrow \quad \frac{dy}{dt} &= -50y \\ \Rightarrow \quad \frac{1}{y}\,dy &= -50\,dt \\ \Rightarrow \quad \int \frac{1}{y}\,dy &= \int (-50)\,dt \\ \Rightarrow \quad \ln|y| &= -50t + C \\ \Rightarrow \quad e^{\ln|y|} &= e^{-50t+C} \\ \Rightarrow \quad |y| &= e^{-50t+C}. \end{aligned}$$

Here, we have a choice for y: either

$$y = e^{-50t+C} \quad \text{or} \quad y = -e^{-50t+C}.$$

For this class, we're always going to want our functions to be non-negative, so we drop the absolute values and write

$$y = e^{-50t+C} = e^{-50t} \underbrace{e^C}_{C \text{ or } C'} = Ce^{-50t}.$$

Hence,

$$\boxed{y = Ce^{-50t}}.$$

2. Find the particular solution to the equation

$$\frac{dA}{dt} = (120 - A) \text{ such that } A(0) = 100,\ A < 120 \text{ for all } t.$$

Solution: The fact that $A < 120$ for all t tells us that $120 - A > 0$. So,

$$\begin{aligned} \frac{dA}{dt} &= 120 - A \\ \Rightarrow \quad \frac{1}{120 - A}\,dA &= dt \\ \Rightarrow \quad \int \frac{1}{120 - A}\,dA &= \int dt \end{aligned}$$

The LHS is a u-sub problem. If $u = 120 - A$, then $du = -\,dA$. So

$$\int \frac{1}{120 - A}\,dA = \int -\frac{1}{u}\,du = -\ln|u| = -\ln|120 - A|.$$

Thus,

$$\begin{aligned} \int \frac{1}{120-A}\,dA &= \int dt \\ \Rightarrow \quad -\ln|120-A| &= t + C \\ \Rightarrow \quad \ln|120-A| &= -t \underbrace{-C}_{+C \text{ or } C'} \end{aligned}$$

We have assumed that $120 - A > 0$, so we may drop the absolute values. Hence,

$$\begin{aligned} \ln|120-A| &= -t + C \\ \Rightarrow \quad \ln(120-A) &= -t + C \\ \Rightarrow \quad e^{\ln(120-A)} &= e^{-t+C} \\ \Rightarrow \quad 120 - A &= e^{-t+C} \\ \Rightarrow \quad 120 - e^{-t+C} &= A \\ \Rightarrow \quad 120 - \underbrace{e^{C}}_{C \text{ or } C''} e^{-t} &= A. \end{aligned}$$

So our solution is of the form

$$A = 120 - Ce^{-t}.$$

We were told that $A(0) = 100$, which means

$$100 = 120 - Ce^{-0} = 120 - C \;\Rightarrow\; C = 20.$$

Therefore,

$$\boxed{A = 120 - 20e^{-t}}.$$

3. Find the particular solution to

$$\frac{dy}{dx} = 9x^2 e^{-x^3}$$

given $y = 7$ when $x = 3$.

Solution: We write

$$\begin{aligned} \frac{dy}{dx} &= 9x^2 e^{-x^3} \\ \Rightarrow \quad dy &= 9x^2 e^{-x^3}\,dx \\ \Rightarrow \quad \int dy &= \int 9x^2 e^{-x^3}\,dx \end{aligned}$$

We use u-substitution to integrate the RHS. Take $u = -x^3$, $\dfrac{du}{dx} = -3x^2$ which gives

$$\int 9x^2 e^{-x^3}\,dx = \int 9x^2 e^u \underbrace{\left(-\frac{du}{3x^2}\right)}_{dx}$$
$$= \int -3e^u\,du$$
$$= -3e^u + C$$
$$= -3e^{-x^3} + C$$

Hence, we see that

$$y = -3e^{-x^3} + C.$$

Now, since $y = 7$ when $x = 3$, we write

$$7 = -3e^{-(3)^3} + C$$
$$\Rightarrow \quad 7 = -3e^{-27} + C$$
$$\Rightarrow \quad 7 + 3e^{-27} = C$$

We conclude

$$y = \boxed{-3e^{-x^3} + 7 + 3e^{-27}}.$$

4. Find the particular solution to

$$\frac{dy}{dt} + y\sin t = 0$$

given $y(\pi) = 8$.

Solution: Separating our variables, we write

$$\frac{dy}{dt} + y\sin t = 0$$
$$\Rightarrow \quad \frac{dy}{dt} = -y\sin t$$
$$\Rightarrow \quad \frac{dy}{y} = -\sin t\,dt$$
$$\Rightarrow \quad \int \frac{dy}{y} = \int -\sin t\,dt$$
$$\Rightarrow \quad \ln|y| = \cos t + C$$
$$\Rightarrow \quad |y| = e^{\cos t + C}$$
$$= e^{\cos t} e^C$$

$$\Rightarrow \quad |y| = Ce^{\cos t}$$

We assume that $y > 0$ and so

$$y = Ce^{\cos t}.$$

We use $y(\pi) = 8$ to solve for C:

$$\begin{aligned} 8 &= Ce^{\cos \pi} \\ &= Ce^{-1} \\ \Rightarrow \quad 8e &= C \end{aligned}$$

Note that

$$(8e)e^{\cos t} = (8e^1)e^{\cos t} = 8e^{\cos t + 1}.$$

Thus,

$$y = \boxed{y = 8e^{\cos t+1}}.$$

5. Find the general solution to

$$\frac{dy}{dt} = 7e^{-5t-y}.$$

Solution: Write

$$\begin{aligned} \frac{dy}{dt} &= 7e^{-5t-y} \\ &= 7e^{-5t}e^{-y} \\ \Rightarrow \quad e^y \frac{dy}{dt} &= 7e^{-5t} \underbrace{e^{-y}e^y}_{e^{-y+y}=e^0=1} \\ \Rightarrow \quad e^y\,dy &= 7e^{-5t}\,dt \\ \Rightarrow \quad \int e^y\,dy &= \int 7e^{-5t}\,dt \\ \Rightarrow \quad e^y &= -\frac{7}{5}e^{-5t} + C \\ \Rightarrow \quad \ln(e^y) &= \ln\left(-\frac{7}{5}e^{-5t} + C\right) \\ \Rightarrow \quad y &= \boxed{\ln\left(-\frac{7}{5}e^{-5t} + C\right)} \end{aligned}$$

6. Find the particular solution to

$$\frac{dy}{dx} = \frac{2x+1}{3y^2}$$

given $y = 4$ when $x = 0$.

Solution: Separating variables,

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x+1}{3y^2} \\ \Rightarrow \quad 3y^2\,dy &= (2x+1)\,dx \\ \Rightarrow \quad \int 3y^2\,dy &= \int (2x+1)\,dx \\ \Rightarrow \quad y^3 &= x^2+x+C \\ \Rightarrow \quad y &= \sqrt[3]{x^2+x+C} \end{aligned}$$

Since $y = 4$ when $x = 0$,

$$\begin{aligned} 4 &= \sqrt[3]{0^2+0+C} \\ &= \sqrt[3]{C} \\ 64 &= C \end{aligned}$$

Hence,

$$y = \boxed{\sqrt[3]{x^2+x+64}}.$$

Lesson 8: Differential Equations: Separation of Variables (II)

1. Solutions to In-Class Examples

Example 1. Find the general solution to

$$x^3y' = y' + x^2e^{-y}.$$

Solution: We first want to move y' to one side and everything else to the other. We have

$$\begin{aligned} x^3y' &= y' + x^2e^{-y} \\ \Rightarrow \quad x^3y' - y' &= x^2e^{-y} \\ \Rightarrow \quad (x^3-1)y' &= x^2e^{-y} \\ \Rightarrow \quad y' &= \frac{x^2e^{-y}}{x^3-1} \\ \Rightarrow \quad \frac{dy}{dx} &= \frac{x^2e^{-y}}{x^3-1} \end{aligned}$$

Now, we want to separate the variables. Multiply both sides by e^y. Then we get

$$\begin{aligned} e^y\,\frac{dy}{dx} &= \frac{x^2e^{-y}e^y}{x^3-1} \\ \Rightarrow \quad e^y\,\frac{dy}{dx} &= \frac{x^2e^{-y+y}}{x^3-1} \\ \Rightarrow \quad e^y\,\frac{dy}{dx} &= \frac{x^2e^0}{x^3-1} \\ \Rightarrow \quad e^y\,\frac{dy}{dx} &= \frac{x^2}{x^3-1} \\ \Rightarrow \quad e^y\,dy &= \frac{x^2}{x^3-1}\,dx. \end{aligned}$$

Our next step is integration, that is,

$$\int e^y\,dy = \int \frac{x^2}{x^3-1}\,dx.$$

The RHS is a u-substitution problem. Let $u = x^3-1$, then $du = 3x^2\,dx \;\Rightarrow\; \dfrac{du}{3x^2} = dx$. So

$$\begin{aligned}\int \frac{x^2}{x^3-1}\,dx &= \int \frac{x^2}{u}\left(\frac{1}{3x^2}\,du\right)\\ &= \int \frac{1}{3u}\,du\\ &= \frac{1}{3}\int \frac{1}{u}\,du\\ &= \frac{1}{3}\ln|u| + C\\ &= \frac{1}{3}\ln|x^3-1| + C.\end{aligned}$$

Thus,

$$\begin{aligned}\int e^y\,dy &= \int \frac{x^2}{x^3-1}\,dx\\ \Rightarrow \quad e^y &= \frac{1}{3}\ln|x^3-1| + C.\end{aligned}$$

Next, we need to get y by itself so apply ln to both sides. We get

$$\begin{aligned}\ln e^y &= \ln\left(\frac{1}{3}\ln|x^3-1| + C\right)\\ \Rightarrow \quad y &= \boxed{\ln\left(\frac{1}{3}\ln|x^3-1| + C\right)}.\end{aligned}$$

NOTE 22. Observe that

$$\ln(a+b) \neq \ln a + \ln b,$$

so the solution is **NOT** the same as

$$\ln\left(\frac{1}{3}\ln|x^3-1|\right) + C.$$

EXAMPLE 2. Suppose during a chemical reaction, a substance is converted into a different substance at a rate inversely proportional to the amount of the original substance at any given time t. If there were initially 10 grams of the original substance and after an hour only 8 grams remained, how much of the original substance is there after 2 hours? Round your answer to the nearest hundredth.

Solution: Let $y(t)$ be the number of grams of the original substance at time t hours. We are told that the rate of change is **inversely proportional** to the amount of the substance. That means our differential equation is given by $\dfrac{dy}{dt} = \dfrac{k}{y}$ where k is an unknown constant that we will need to find.

Ultimately, the question asks us to find $y(2)$ given the following information:

$$\frac{dy}{dt} = \frac{k}{y}$$
$$y(0) = 10$$
$$y(1) = 5.$$

We see our differential equation is separable, so we write

$$\frac{dy}{dt} = \frac{k}{y}$$
$$\Rightarrow \quad y\,dy = k\,dt.$$

Next, we integrate:

$$\int y\,dy = \int k\,dt$$
$$\Rightarrow \quad \frac{1}{2}y^2 = kt + C.$$

Because we are told $y(0) = 10$, we have

$$\frac{1}{2}(10)^2 = k(0) + C \;\Rightarrow\; C = 50.$$

So our equation becomes

$$\frac{1}{2}y^2 = kt + 50.$$

Moreover, since $y(1) = 8$,

$$\frac{1}{2}(8)^2 = k(1) + 50$$
$$\Rightarrow \quad 32 = k + 50$$
$$\Rightarrow \quad 32 - 50 = k$$
$$\Rightarrow \quad -18 = k.$$

Solving for y,

$$\frac{1}{2}y^2 = -18t + 50$$
$$\Rightarrow \quad y^2 = -36t + 100$$
$$\Rightarrow \quad y = \sqrt{-36t + 100}.$$

We are asked to find $y(2)$, so

$$y(2) = \sqrt{-36(2) + 100}$$
$$= \sqrt{-72 + 100}$$

$$= \sqrt{28}$$

$$\approx \boxed{5.29 \text{ grams}}$$

EXAMPLE 3. A 500-gallon tank initially contains 250 gallons of brine, a salt and water combination. Brine containing 2 pounds of salt per gallon flows into the tank at a rate of 4 gallons per minute. Suppose the well-stirred mixture flows out of the tank at a rate of 2 gallons per minute. Set up a differential equation for the amount of salt (in pounds) in the tank at time t (minutes).

Solution: We are asked to set up the equation but not to solve it. For this type of problem, units are very important. Let $A(t)$ be the pounds of salt in the tank at time t minutes. By how we have defined our function, the units associated to $\frac{dA}{dt}$ are lbs/min. Hence

$$\frac{dA}{dt} = [\text{Rate of salt in lbs/min}] - [\text{Rate of salt out lbs/min}].$$

[Rate of salt in]: Every minute, 4 gallons flow into the tank and each gallon contains 2 pounds of salt. So

$$[\text{Rate of salt in}] = \underbrace{\left(\frac{2 \text{ lbs}}{1 \text{ gal}}\right)}_{\substack{\text{salt in} \\ \text{per gal}}} \underbrace{\left(\frac{4 \text{ gal}}{1 \text{ min}}\right)}_{\substack{\text{water in} \\ \text{per min}}} = \text{salt in per minute}$$

$$= 8 \text{ lbs/min}$$

[Rate of salt out]: The difficulty here is understanding what well-stirred means. Well-stirred means that each gallon in the tank has as much salt in it as any other gallon. Thus, to correctly interpret "well-stirred", we need to take the *total* amount of salt in the tank and divide it by the *total* amount of liquid in the tank. There are initially 250 gallons of brine and each minute, 2 gallons are added to the tank (to see that 2 gallons are added to the tank each minute, consider

$$\underset{\substack{\uparrow \\ \text{in}}}{4 \text{ gal/min}} - \underset{\substack{\uparrow \\ \text{out}}}{2 \text{ gal/min}} = +2 \text{ gal/min}).$$

So, we see that

$$[\text{Rate of salt out}] = \underbrace{\left(\frac{A(t) \text{ lbs}}{250 + (4-2)t \text{ gal}}\right)}_{\substack{\text{salt out} \\ \text{per gal}}} \underbrace{\left(\frac{2 \text{ gal}}{1 \text{ min}}\right)}_{\substack{\text{water out} \\ \text{per min}}} = \text{salt out per minute}$$

$$= \frac{2A(t)}{250 + 2t} \text{ lbs/min}.$$

Therefore, our differential equation is

$$\frac{dA}{dt} = [\text{Rate of salt in}] - [\text{Rate of salt out}] = \boxed{8 - \frac{2A(t)}{250 + 2t}}.$$

NOTE 23. This is **not** a separable equation. To actually solve for this $A(t)$, we need a different technique which we will cover in Lesson 9. Fortunately, we are only asked to set up the differential equation.

EXAMPLE 4. A 700-gallon tank initially contains 400 gallons of brine containing 50 pounds of dissolved salt. Brine containing 6 pounds of salt per gallon flows into the tank at a rate of 3 gallons per minute, and the well-stirred mixture flows out of the tank at a rate of 3 gallons per minute. Find the amount of salt in the tank after 10 minutes. Round your answer to 3 decimal places.

Solution: Let $A(t)$ be pounds of salt in the tank at time t minutes where we are given $A(0) = 50$. Our goal is to find $A(10)$.

As before, we have

$$\frac{dA}{dt} = [\text{Rate of salt in lbs/min}] - [\text{Rate of salt out lbs/min}].$$

[Rate of salt in]:

$$\underbrace{\left(\frac{6 \text{ lbs}}{1 \text{ gal}}\right)}_{\substack{\text{salt in}\\\text{per gal}}}\underbrace{\left(\frac{3 \text{ gal}}{1 \text{ min}}\right)}_{\substack{\text{water in}\\\text{per min}}} = 18 \text{ lbs/min} = \text{salt in per minute.}$$

[Rate of salt out]:

$$\underbrace{\left(\frac{A(t) \text{ lbs}}{400 - (3-3)t \text{ gal}}\right)}_{\substack{\text{salt out}\\\text{per gal}}}\underbrace{\left(\frac{3 \text{ gal}}{1 \text{ min}}\right)}_{\substack{\text{water out}\\\text{per min}}} = \frac{3A(t)}{400} \text{ lbs/min} = \text{salt out per minute.}$$

Hence, our differential equation becomes

$$\frac{dA}{dt} = \underset{\substack{\uparrow\\\text{in}}}{18} - \underset{\substack{\uparrow\\\text{out}}}{\frac{3A(t)}{400}} = \frac{7200 - 3A}{400}.$$

This is a separable equation.

NOTE 24. The difference between this example and Example 3 is that the amount of water coming into the tank and the amount of water leaving the tank cancel each other out and this eliminates the t on the RHS. This transforms the problem into a separation of variables differential equation.

We write

$$\left(\frac{400}{7200 - 3A}\right) dA = dt$$

$$\Rightarrow \quad \int \left(\frac{400}{7200-3A}\right) dA = \int dt$$

To integrate the LHS, take $u = 7200 - 3A$, then $\dfrac{du}{dA} = -3$. Then,

$$\begin{aligned}
\int \left(\frac{400}{7200-3A}\right) dA &= \int \left(\frac{400}{u}\right) \underbrace{\left(-\frac{du}{3}\right)}_{dA} \\
&= \int -\frac{400}{3u}\, du \\
&= -\frac{400}{3} \ln|u| \\
&= -\frac{400}{3} \ln|7200 - 3A| \\
&= -\frac{400}{3} \ln(7200 - 3A)
\end{aligned}$$

since we assume that $7200 - 3A > 0$. Hence,

$$\begin{aligned}
& \int \left(\frac{400}{7200-3A}\right) dA = \int dt \\
\Rightarrow \quad & -\frac{400}{3} \ln(7200 - 3A) = t + C \\
\Rightarrow \quad & \ln(7200 - 3A) = -\frac{3t}{400} + C \\
\Rightarrow \quad & 7200 - 3A = e^{3t/400+C} \\
\Rightarrow \quad & 7200 - 3A = e^{3t/400} e^{C} \\
\Rightarrow \quad & 7200 - 3A = Ce^{3t/400} \\
\Rightarrow \quad & 7200 + Ce^{3t/400} = 3A \\
\Rightarrow \quad & 2400 + Ce^{3t/400} = A
\end{aligned}$$

We conclude that

$$A = Ce^{3t/400} + 2400.$$

Since we are given $A(0) = 50$,

$$50 = C \underbrace{e^{-3(0)/400}}_{1} + 2400 = C + 2400 \Rightarrow C = -2350.$$

Thus,

$$A = -2350e^{-3t/400} + 2400.$$

So,

$$A(10) = -2350e^{-30/400} + 2400 \approx \boxed{219.803 \text{ lbs}}.$$

2. Additional Examples

EXAMPLES.

1. Find $y(2)$ if y is a function of x such that

$$xy^6y' = 2 \text{ and } y = 1 \text{ when } x = 1.$$

Solution: This is a separable function, so our first step should be to move x to the RHS. Write

$$\begin{aligned} xy^6y' &= 2 \\ \Rightarrow \quad y^6y' &= \frac{2}{x} \\ \Rightarrow \quad y^6\frac{dy}{dx} &= \frac{2}{x} \\ \Rightarrow \quad y^6\,dy &= \frac{2}{x}\,dx. \end{aligned}$$

Now, we integrate:

$$\begin{aligned} \int y^6 dy &= \int \frac{2}{x}\,dx \\ \Rightarrow \quad \frac{1}{7}y^7 &= 2\ln|x| + C. \end{aligned}$$

We are given that $y = 1$ when $x = 1$, so to find C we can write:

$$\begin{aligned} \frac{1}{7}y^7 &= 2\ln|x| + C \\ \Rightarrow \quad \frac{1}{7}(1)^7 &= 2\underbrace{\ln|1|}_{0} + C \\ \Rightarrow \quad C &= \frac{1}{7} \end{aligned}$$

Hence,

$$\frac{1}{7}y^7 = 2\ln|x| + \frac{1}{7}.$$

The question asks us to find $y(2)$. So we need to get y by itself first. Write

$$\begin{aligned} \frac{1}{7}y^7 &= 2\ln|x| + \frac{1}{7} \\ \Rightarrow \quad y^7 &= 14\ln|x| + 1 \\ \Rightarrow \quad y &= \sqrt[7]{14\ln|x| + 1}. \end{aligned}$$

Thus,

$$y(2) = \boxed{\sqrt[7]{14\ln(2)+1}}.$$

2. Find the general solution to

$$t^2 y' + 3y = 0.$$

Solution: We separate the variables:

$$\begin{aligned} t^2 y' + 3y &= 0 \\ \Rightarrow \quad t^2 \frac{dy}{dt} &= -3y \\ \Rightarrow \quad \frac{1}{y}\,dy &= -\frac{3}{t^2}\,dt \end{aligned}$$

Next, we integrate.

$$\begin{aligned} \int \frac{1}{y}\,dy &= \int -\frac{3}{t^2}\,dt = \int -3t^{-2}\,dt \\ \Rightarrow \quad \ln y &= -3\left(\frac{1}{-2+1}\right)t^{-2+1} + C \\ &= -3\left(\frac{1}{-1}\right)t^{-1} + C \\ &= \frac{3}{t} + C \\ \Rightarrow \quad y &= e^{3/t+C} \\ &= \boxed{Ce^{3/t}} \end{aligned}$$

3. Find the particular solution to

$$t^2 \frac{dy}{dt} + y = 0$$

given $y(3) = 9$ where $t > 0$.

Solution: We separate the variables:

$$\begin{aligned} t^2 \frac{dy}{dt} + y &= 0 \\ \Rightarrow \quad t^2 \frac{dy}{dt} &= -y \\ \Rightarrow \quad \frac{dy}{y} &= -\frac{1}{t^2}\,dt \\ \Rightarrow \quad \int \frac{dy}{y} &= \int -\frac{1}{t^2}\,dt \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \ln|y| &= \frac{1}{t} + C \\
\Rightarrow \quad |y| &= e^{1/t+C} \\
&= e^{1/t}e^{C} \\
|y| &= Ce^{1/t}
\end{aligned}$$

We assume that $y > 0$. Thus,

$$y = Ce^{1/t}.$$

We use $y(3) = 9$ to solve for C. Write

$$\begin{aligned}
9 &= Ce^{1/3} \\
\Rightarrow \quad \frac{9}{e^{1/3}} &= C \\
\Rightarrow \quad 9e^{-1/3} &= C
\end{aligned}$$

Note that

$$(9e^{-1/3})e^{1/t} = 9e^{1/t-1/3}.$$

Hence,

$$y = \boxed{9e^{1/t-1/3}}.$$

4. Find the particular solution to

$$\frac{dy}{dx} = \frac{\cos(11x)}{e^{11y}}$$

given $y = 2$ when $x = 0$.

Solution: Write

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\cos(11x)}{e^{11y}} \\
\Rightarrow \quad e^{11y}\,dy &= \cos(11x)\,dx \\
\Rightarrow \quad \int e^{11y}\,dy &= \int \cos(11x)\,dx \\
\Rightarrow \quad \frac{1}{11}e^{11y} &= \frac{1}{11}\sin(11x) + C \\
\Rightarrow \quad e^{11y} &= \sin(11x) + C \\
\Rightarrow \quad 11y &= \ln(\sin(11x) + C) \\
\Rightarrow \quad y &= \frac{1}{11}\ln(\sin(11x) + C)
\end{aligned}$$

Solving for C, we use $y = 2$ when $x = 0$:

$$\begin{aligned} 2 &= \frac{1}{11}\ln(\underbrace{\sin(11\cdot 0)}_{0}+C) \\ &= \frac{1}{11}\ln(C) \\ \Rightarrow \quad 22 &= \ln(C) \\ \Rightarrow \quad e^{22} &= C \end{aligned}$$

Hence,

$$y = \boxed{\frac{1}{11}\ln(\sin(11x)+e^{22})}.$$

5. Find the general solution to

$$\frac{dy}{dx} = \sqrt{4y}e^{x+8}.$$

Solution: Separating variables,

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{4y}e^{x+8} \\ \Rightarrow \quad \frac{dy}{\sqrt{4y}} &= e^{x+8}\,dx \\ \Rightarrow \quad (4y)^{-1/2}\,dy &= e^{x+8}\,dx \\ \Rightarrow \quad \int (4y)^{-1/2}\,dy &= \int e^{x+8}\,dx \end{aligned}$$

Now, to integrate the LHS, take $u = 4y$, $\dfrac{du}{dy} = 4$ and write

$$\begin{aligned} \int (4y)^{-1/2}\,dy &= \int u^{-1/2}\underbrace{\left(\frac{du}{4}\right)}_{dy} \\ &= \frac{1}{4}\left(\frac{1}{-1/2+1}\right)u^{-1/2+1}\,du \\ &= \frac{1}{4}\left(\frac{1}{1/2}\right)u^{1/2} \\ &= \frac{1}{4}(2)u^{1/2} \\ &= \frac{1}{2}u^{1/2} \\ &= \frac{1}{2}(4y)^{1/2} \end{aligned}$$

Putting this together,

$$\begin{aligned}
\int (4y)^{-1/2}\,dy &= \int e^{x+8}\,dx \\
\Rightarrow \quad \frac{1}{2}(4y)^{1/2} &= e^{x+8} + C \\
\Rightarrow \quad (4y)^{1/2} &= 2e^{x+8} + C \\
\Rightarrow \quad 4y &= (2e^{x+8} + C)^2 \\
\Rightarrow \quad y &= \boxed{\frac{1}{4}(2e^{x+8} + C)^2}
\end{aligned}$$

Lesson 9: First Order Linear Differential Equations (I)

1. First Order Linear Differential Equations

Today we introduce a new type of differential equation called a **first order linear differential equation (FOLDE)** and describe the method by which we solve them.

FOLDE are of the form

$$\frac{dy}{dt} + P(t)y = Q(t) \tag{5}$$

Observe the following about equation (5):

(a) the "+" is very important. Any "−" must be included in the $P(t)$

(b) the y being multiplied by the $P(t)$ is to the first power

(c) $P(t)$ and $Q(t)$ do **not** include any y, they are functions only of t

If you cannot write the differential equation *exactly* in the form of equation (5), then it is **not** a FOLDE.

Ex 1.

- $\frac{dy}{dt} + \frac{3y}{t} = t^2$ is a FOLDE with $P(t) = \frac{3}{t}$ and $Q(t) = t^2$
- $y' - \frac{1}{x}y = \sin x^2$ is a FOLDE with $P(x) = -\frac{1}{x}$ and $Q(x) = \sin x^2$

Note 25 (FOLDE vs Separation of Variables). FOLDE is related to, but different, from separation of variables. Sometimes these methods overlap but most of the time only one method will apply. Take time to practice identifying when a particular method will apply.

The key to solving this type of differential equation is an **integrating factor**, that is, the function

$$u(t) = e^{\int P(t)\,dt}. \tag{6}$$

The **solution** of a FOLDE is found via

$$y \cdot u(t) = \int Q(t)u(t)\,dt. \tag{7}$$

y is the actual solution so, after we integrate the RHS, we must divide through by $u(t)$.

To apply the FOLDE method, take the following steps:

1. Find P, Q

2. Find the integrating factor

3. Set up equation (7)

EXAMPLES.

1. Find the general solution to

$$\frac{dy}{dx} + \frac{y}{x} = x.$$

Solution: We go through our steps.

Step 1: Find P, Q

$$P(x) = \frac{1}{x}, \quad Q(x) = x$$

Step 2: Find the integrating factor

$$\begin{aligned} u(x) &= e^{\int P(x)\,dx} \\ &= e^{\int \frac{1}{x}\,dx} \\ &= e^{\ln|x|} \\ &= |x| \overset{(\star)}{=} x \end{aligned}$$

NOTE 26 ($\star$). We are cheating here and using the simplifying assumption that $x > 0$, which would mean that $|x| = x$.

Step 3: Set up equation (7)

$$\begin{aligned} y \cdot u(x) &= \int Q(x)u(x)\,dx \\ \Rightarrow \quad y \underset{u(x)}{\overset{\uparrow}{x}} &= \int \underbrace{(x)}_{Q(x)} \cdot \underbrace{(x)}_{u(x)}\,dx \\ &= \int x^2\,dx \\ &= \frac{1}{3}x^3 + C \\ yx &= \frac{1}{3}x^3 + C \end{aligned}$$

We divide both sides by x to get

$$\boxed{y = \frac{1}{3}x^2 + \frac{C}{x}}.$$

NOTE 27. It is important that when you divide through by x, you **also** divide the C as well. C absorbs constants, **not** functions.

2. Given $t^2y' + ty = 25$ and $y(1) = 0$, find $y(6)$.

Solution: This is a FOLDE but it isn't in the right form (we want it to look like equation (5)). Write

$$\begin{aligned} t^2y' + ty &= 25 \\ \Rightarrow \quad y' + \frac{ty}{t^2} &= \frac{25}{t^2} \\ \Rightarrow \quad y' + \frac{y}{t} &= \frac{25}{t^2} \end{aligned}$$

Now, since our new equation looks like equation (5), we can proceed.

Step 1: Find P, Q

$$P(t) = \frac{1}{t}, \quad Q(t) = \frac{25}{t^2}$$

Step 2: Find the integrating factor

$$\begin{aligned} u(t) &= e^{\int P(t)\,dt} \\ &= e^{\int \frac{1}{t}\,dt} \\ &= e^{\ln|t|} \\ &= |t| \overset{(\star\star)}{=} t \end{aligned}$$

$(\star\star)$ Again, we cheat and assume that $t > 0$.

Step 3: Set up equation (7)

$$\begin{aligned} y \cdot u(t) &= \int Q(t)u(t)\,dt \\ \Rightarrow \quad y \underset{u(t)}{\underset{\uparrow}{t}} &= \int \underbrace{\left(\frac{25}{t^2}\right)}_{Q(t)} \cdot \underset{u(t)}{\underset{\uparrow}{(t)}}\ dt \\ \Rightarrow \quad y \cdot t &= \int \frac{25}{t}\,dt \\ &= 25\ln|t| + C \\ &= 25\ln(t) + C \text{ since } t > 0 \end{aligned}$$

So we have

$$y \cdot t = 25\ln(t) + C$$

which means

$$y = \frac{25\ln(t)}{t} + \frac{C}{t}.$$

We have found the *general* solution, but we were asked to find $y(6)$ given $y(1) = 0$.

Since $y(1) = 0$,

$$0 = \underbrace{\frac{25\ln(1)}{1}}_{0} + \frac{C}{1} \;\Rightarrow\; C = 0.$$

Hence,

$$y = \frac{25\ln(t)}{t}.$$

Finally,

$$\boxed{y(6) = \frac{25\ln(6)}{6}}$$

3. Find the general solution to

$$y' + 4(\tan 4x)y = 6\cos 4x$$

such that $0 < x < \frac{\pi}{8}$.

Solution: This is in the proper form so we can apply our method.

Step 1: Find P, Q

$$P(x) = 4\tan 4x, \quad Q(t) = 6\cos 4x$$

Step 2: Find the integrating factor

To integrate $P(x)$, we need to do a substitution but we want to reserve u to denote the integrating factor. Let

$$P(x) = 4\tan 4x = \frac{4\sin 4x}{\cos 4x}$$

and take $w = \cos 4x$, $\frac{dw}{dx} = -4\sin 4x$. So we write

$$\begin{aligned}
\int P(x)\,dx &= \int \frac{4\sin 4x}{\cos 4x}\,dx \\
&= \int \frac{4\sin 4x}{w} \underbrace{\left(-\frac{dw}{4\sin 4x}\right)}_{dx} \\
&= \int -\frac{1}{w}\,dw \\
&= -\ln|w| \\
&= -\ln|\cos 4x| \\
&= \ln|\sec 4x| \text{ since } (\cos 4x)^{-1} = \sec 4x
\end{aligned}$$

Thus, our integrating factor is given by

$$u(x) = e^{\int P(x)\,dx} = e^{\ln|\sec 4x|} = |\sec 4x|.$$

Over the interval $0 < x < \frac{\pi}{8}$, $\sec 4x > 0$ so we may drop the absolute values.

Step 3: Set up equation (7)

$$\begin{aligned}
y \cdot u(x) &= \int Q(x) \cdot u(x)\,dx \\
\Rightarrow \quad y \underbrace{\sec 4x}_{u(x)} &= \int \underbrace{6\cos 4x}_{Q(x)} \underbrace{(\sec 4x)}_{u(x)}\,dx \\
\Rightarrow \quad y \sec 4x &= \int 6\,dx \\
\Rightarrow \quad y \sec 4x &= 6x + C \\
\Rightarrow \quad y &= \frac{6x}{\sec 4x} + \frac{C}{\sec 4x} \\
&= \boxed{6x\cos 4x + C\cos 4x}
\end{aligned}$$

4. Find the general solution to

$$\frac{dy}{dt} + e^t y = -25e^t.$$

Solution: This is in the correct form so we go through our method.

Step 1: Find P, Q

$$P(t) = e^t, \quad Q(t) = -25e^t$$

Step 2: Find the integrating factor

$$u(t) = e^{\int P(t)\,dt} = e^{\int e^t\,dt} = e^{e^t}$$

Step 3: Set up equation (7)

$$\begin{aligned}
y \cdot u(t) &= \int Q(t)u(t)\,dt \\
\Rightarrow \quad y \underbrace{e^{e^t}}_{u(t)} &= \int \underbrace{-25e^t}_{Q(t)} \underbrace{e^{e^t}}_{u(t)}\,dt \\
\Rightarrow \quad y e^{e^t} &= \underbrace{\int -25e^t e^{e^t}\,dt}_{(\star\star\star)}
\end{aligned}$$

where $(\star\star\star)$ is a u-substitution problem.

For $(\star\star\star)$, take $w = e^t$, $\dfrac{dw}{dt} = e^t$ and write

$$\int -25e^t e^{e^t}\,dt = \int -25e^t e^w \underbrace{\left(\frac{dw}{e^t}\right)}_{dt}$$
$$= \int -25e^w\,dw$$
$$= -25e^w + C$$
$$= -25e^{e^t} + C$$

Thus,

$$ye^{e^t} = \underbrace{\int -25e^t e^{e^t}\,dt}_{(\star\star\star)}$$
$$\Rightarrow \quad ye^{e^t} = -25e^{e^t} + C$$
$$\Rightarrow \quad y = \frac{-25e^{e^t} + C}{e^{e^t}}$$
$$= \boxed{-25 + Ce^{-e^t}}$$

2. Additional Examples

EXAMPLES.

1. Find the general solution to

$$-\frac{dy}{dt} + y = t.$$

Solution: Again, this is not in the same form as equation (5). So we rewrite:

$$-\frac{dy}{dt} + y = t \;\Rightarrow\; \frac{dy}{dt} - y = -t.$$

Now, we can apply our method.

Step 1: Find P, Q

$$P(t) = -1, \;\; Q(t) = -t$$

Step 2: Find the integrating factor

$$u(t) = e^{\int P(t)\,dt}$$
$$= e^{\int -1\,dt}$$
$$= e^{-t}$$

Step 3: Set up equation (7)

$$y \cdot u(t) = \int Q(t)u(t)\,dt$$

$$\Rightarrow \quad y \underbrace{e^{-t}}_{u(t)} = \int \underbrace{(-t)}_{Q(t)} \cdot \underbrace{(e^{-t})}_{u(t)}\,dt$$

$$\Rightarrow \quad ye^{-t} = \underbrace{\int -te^{-t}\,dt}_{\substack{\text{integration by parts} \\ (\diamond)}}$$

We find $(\diamond)$ using integration by parts. By LIATE,

$$u = -t \qquad dv = e^{-t}\,dt$$

$$du = -dt \qquad v = \int e^{-t}\,dt = -e^{-t}$$

So,

$$\int -te^{-t}\,dt = \underbrace{-t}_{u}\underbrace{(-e^{-t})}_{v} - \int \underbrace{(-e^{-t})}_{v}\underbrace{(-dt)}_{du}$$

$$= te^{-t} - \int e^{-t}\,dt$$

$$= te^{-t} + e^{-t} + C.$$

Therefore, we have

$$ye^{-t} = \int -te^{-t}\,dt \overset{(\diamond)}{=} te^{-t} + e^{-t} + C$$

which means

$$\boxed{y = t + 1 + Ce^t}.$$

Observe that the only unknown constant here is C. The 1 is **not** absorbed by an arbitrary constant because it is not arbitrary.

2. Find the general solution to

$$(y-1)\sin x\,dx - dy = 0.$$

Solution: This equation is both a FOLDE and separable. We find the solution using the FOLDE method. However, it is certainly not in the proper form for either method. We write

$$(y-1)\sin x\,dx - dy = 0$$

$$\Rightarrow \quad (y-1)\sin x\,dx = dy$$

$$\Rightarrow \quad (y-1)\sin x = \frac{dy}{dx}$$

$$\Rightarrow \quad y\sin x - \sin x = \frac{dy}{dx}$$

$$\Rightarrow \quad -\sin x = \frac{dy}{dx} - (\sin x)y$$

Now this is in the form as given in equation (5).

Step 1: Find P, Q

$$P(x) = -\sin x, \quad Q(x) = -\sin x$$

Step 2: Find the integrating factor

$$u(x) = e^{\int P(x)\,dx} = e^{\int -\sin x\,dx} = e^{\cos x}$$

Step 3: Set up equation (7)

$$y \cdot u(x) = \int Q(x)u(x)\,dx$$

$$\Rightarrow \quad y\underbrace{e^{\cos x}}_{u(x)} = \int \underbrace{(-\sin x)}_{Q(x)}\underbrace{(e^{\cos x})}_{u(x)}\,dx$$

$$\Rightarrow \quad ye^{\cos x} = \underbrace{\int -(\sin x)e^{\cos x}\,dx}_{\substack{u\text{-sub} \\ (\diamond\diamond)}}$$

We find $(\diamond\diamond)$. Let $u = \cos x$, then $du = -\sin x$. So

$$\int -(\sin x)e^{\cos x}\,dx = \int e^u\,du$$
$$= e^u + C$$
$$= e^{\cos x} + C$$

We write

$$ye^{\cos x} = e^{\cos x} + C$$

which means

$$\boxed{y = 1 + \frac{C}{e^{\cos x}}}$$

Again, the only unknown here is C and you may **not** replace the 1 with an arbitrary constant because it is not arbitrary.

3. Find the general solution to

$$\frac{dy}{dx} + 11y = 13.$$

Solution: Since this is in the correct form, we apply our method.

Step 1: Find P, Q

$$P(x) = 11, \quad Q(x) = 13$$

Step 2: Find the integrating factor

$$\begin{aligned} u(x) &= e^{\int P(x)\,dx} \\ &= e^{\int 11\,dx} \\ &= e^{11x} \end{aligned}$$

Step 3: Set up equation (7)

$$\begin{aligned} y \cdot u(x) &= \int Q(x)u(x)\,dx \\ \Rightarrow \quad y\underbrace{e^{11x}}_{u(x)} &= \int \underbrace{13}_{Q(x)}\underbrace{\left(e^{11x}\right)}_{u(x)}\,dx \\ \Rightarrow \quad ye^{11x} &= \int 13e^{11x}\,dx \\ \Rightarrow \quad ye^{11x} &= \frac{13}{11}e^{11x} + C \\ \Rightarrow \quad y &= \frac{13}{11} + \frac{C}{e^{11x}} \\ &= \boxed{\frac{13}{11} + Ce^{-11x}} \end{aligned}$$

4. Find the general solution to

$$3x^2y + x^3y' = 7\sec^2 x \tan x.$$

Solution: This differential equation is not in the proper form to apply our method, so we make some adjustments.

$$\begin{aligned} 3x^2y + x^3y' &= 7\sec^2 x \tan x \\ \Rightarrow \quad x^3y' + 3x^2y &= 7\sec^2 x \tan x \\ \Rightarrow \quad y' + \frac{3x^2}{x^3}y &= \frac{7\sec^2 x \tan x}{x^3} \\ \Rightarrow \quad y' + \frac{3}{x}y &= \frac{7\sec^2 x \tan x}{x^3} \end{aligned}$$

Now, we can go through our steps.

Step 1: Find P, Q

$$P(x) = \frac{3}{x}, \quad Q(x) = \frac{7\sec^2 x \tan x}{x^3}$$

Step 2: Find the integrating factor

$$\begin{aligned} u(x) &= e^{\int P(x)\,dx} \\ &= e^{\int (3/x)\,dx} \\ &= e^{3\ln|x|} \\ &= e^{\ln|x^3|} \\ &= |x^3| = x^3 \end{aligned}$$

where we assume that $x > 0$.

Step 3: Set up equation (7)

$$\begin{aligned} y \cdot u(x) &= \int Q(x)u(x)\,dx \\ \Rightarrow \quad y \underset{\underset{u(x)}{\uparrow}}{x^3} &= \int \underbrace{\left(\frac{7\sec^2 x \tan x}{x^3}\right)}_{Q(x)} \underset{\underset{u(x)}{\uparrow}}{x^3}\ dx \\ \Rightarrow \quad x^3 y &= \int 7\sec^2 x \tan x\,dx \end{aligned}$$

To integrate the RHS, we need to make a couple of observations. First,

$$7\sec^2 x \tan x = 7\sec x \sec x \tan x = 7\sec x(\sec x \tan x).$$

Second,

$$\frac{d}{dx}\sec x = \sec x \tan x.$$

Hence, to compute

$$\int 7\sec^2 x \tan x\,dx = \int 7\sec x(\sec x \tan x)\,dx,$$

we need to take $w = \sec x$, $\dfrac{dw}{dx} = \sec x \tan x$ and write

$$\begin{aligned} \int 7\sec(\sec x \tan x)\,dx &= \int 7w(\sec x \tan x)\ \underbrace{\left(\frac{dw}{\sec x \tan x}\right)}_{dx} \\ &= \int 7w\,dw \\ &= \frac{7}{2}w^2 + C \\ &= \frac{7}{2}\sec^2 x + C \end{aligned}$$

Therefore,

$$
\begin{aligned}
x^3 y &= \int 7\sec^2 x \tan x \, dx \\
\Rightarrow \quad x^3 y &= \frac{7}{2}\sec^2 x + C \\
\Rightarrow \quad y &= \boxed{\frac{7\sec^2 x}{2x^3} + \frac{C}{x^3}}
\end{aligned}
$$

Lesson 10: First Order Linear Differential Equation (II)

1. Solutions to In-Class Examples

Example 1. Suppose a silo contains 50 tons of grain and that a farmer is moving the grain to another silo. If the amount of grain in the second silo changes at a rate proportional to the amount of grain in the first silo, find a differential equation that represents this situation.

Solution: Let $y(t)$ be the amount of grain in the second silo. The amount of grain in the first silo is given by $50 - y(t)$. Hence, our differential equation becomes

$$\boxed{\frac{dy}{dt} = k(50 - y)}.$$

Note 28. Write $k * (50 - y)$ in Loncapa.

Example 2. A store has a storage capacity for 50 printers. If the store currently has 25 printers in inventory and the management determines they sell the printers at a daily rate equal to 10% of the available capacity, when will the store sell out of printers?

Solution: Let $N(t)$ be the number of printers in the store's inventory at t days. The available capacity of printers is given by $50 - N$ (which is the total capacity minus the amount of printers actually in the store). Moreover, our proportionality constant is $k = -.10$ (this is negative because we want to sell printers until there are none left which is to say the number of printers is decreasing). Our differential equation is given by

$$\frac{dN}{dt} = -.10(50 - N).$$

This is a separation of variables problem. We need to get N all on one side and t all on the other. Write

$$\frac{1}{50 - N}\, dN = -.10\, dt$$

$$\Rightarrow \quad \int \frac{1}{50 - N}\, dN = \int -.10\, dt$$

$$\Rightarrow \quad -\ln|50 - N| = -.10t + C$$

$$\Rightarrow \quad \ln|50 - N| = .10t \underbrace{-C}_{C \text{ or } C'}$$

$$\Rightarrow \quad e^{\ln|50-N|} = e^{.10t + C}$$

$$\Rightarrow \quad |50 - N| = Ce^{.10t}$$

Now, since we are assuming that we will never have more printers than the store's capacity, $50 - N \geq 0$. So

$$50 - N = Ce^{.10t} \;\Rightarrow\; 50 - Ce^{.10t} = N.$$

We are told that the store originally has 25 printers, which means $N(0) = 25$. Thus,

$$25 = N(0) = 50 - Ce^{.10(0)} = 50 - C \;\Rightarrow\; C = 25.$$

So,

$$N(t) = 50 - 25e^{.10t}.$$

The question asks us to find when the store will sell out of printers, which is to say we need to find the t such that $N(t) = 0$. Write

$$\begin{aligned} 0 &= 50 - 25e^{.10t} \\ \Rightarrow \quad -50 &= -25e^{.10t} \\ \Rightarrow \quad 2 &= e^{.10t} \\ \Rightarrow \quad \ln 2 &= \ln e^{.10t} \\ \Rightarrow \quad \ln 2 &= .10t \\ \Rightarrow \quad 10\ln 2 &= t. \end{aligned}$$

Therefore, our answer is

$$\boxed{t = 10\ln 2 \text{ days}} \approx 6.931 \text{ days}.$$

Example 3. An 850-gallon tank initially contains 250 gallons of brine containing 50 pounds of dissolved salt. Brine containing 4 pounds of salt per gallon flows into the tank at a rate of 5 gallons per minute. The well-stirred mixture then flows out of the tank at a rate of 2 gallons per minute. How much salt is in the tank when it is full? (Round your answer to the nearest hundredth.)

Solution: Let $A(t)$ be the pounds of salt in the tank at time t minutes. Then

$$\frac{dA}{dt} = [\text{Rate of salt in}] - [\text{Rate of salt out}].$$

[Rate of salt in]: $\left(\dfrac{4 \text{ lbs}}{1 \text{ gal}}\right)\left(\dfrac{5 \text{ gal}}{1 \text{ min}}\right) = 20 \text{ lbs/min}$

[Rate of salt out]: $\left(\dfrac{A(t) \text{ lbs}}{250 + (5-2)t \text{ gal}}\right)\left(\dfrac{2 \text{ gal}}{1 \text{ min}}\right) = \dfrac{2A(t)}{250 + 3t} \text{ lbs/min}$

Hence, our differential equation is

$$\frac{dA}{dt} = 20 - \frac{2A(t)}{250 + 3t}.$$

To find the solution, we use the FOLDE method. However, our differential equation isn't quite in the correct form. So we write

$$\frac{dA}{dt} = 20 - \frac{2A(t)}{250+3t} \;\Rightarrow\; \frac{dA}{dt} + \frac{2A}{250+3t} = 20$$
$$\Rightarrow\; \frac{dA}{dt} + \left(\frac{2}{250+3t}\right) A = 20.$$

Now, since this is now in the right form we can go through the steps.

Step 1: Find P, Q

$$P(t) = \frac{2}{250+3t}, \quad Q(t) = 20$$

Step 2: Find the integrating factor

Recall the integrating factor is given by $u(t) = e^{\int P(t)\,dt}$.

We need to integrate $P(t)$. Write

$$\int P(t)\,dt = \int \frac{2}{250+3t}\,dt$$
$$= \frac{2}{3}\ln|250+3t|$$
$$= \ln(250+3t)^{2/3}.$$

Thus

$$u(t) = e^{\ln(250+3t)^{2/3}} = (250+3t)^{2/3}.$$

Step 3: Set up equation (7)

$$A \cdot u(t) = \int Q(t)u(t)\,dt.$$

Plugging in what we know, we have

$$A\underbrace{(250+3t)^{2/3}}_{u(t)} = \int \underbrace{20}_{Q(t)}\underbrace{(250+3t)^{2/3}}_{u(t)}\,dt = \underbrace{\left(\frac{3}{5}\right)\left(\frac{20}{3}\right)}_{4}(250+3t)^{5/3} + C.$$

Dividing both sides by $u(t) = (250+3t)^{2/3}$,

$$A = 4(250+3t) + \frac{C}{(250+3t)^{2/3}}.$$

We need to find C. We are told that the initial amount of salt in the tank is 50 pounds, so

$$50 = A(0) = 4(250) + \frac{C}{250^{2/3}} \;\Rightarrow\; C = -250^{2/3}(950).$$

We leave this in the exact form and our equation becomes

$$A(t) = 4(250+3t) - \frac{(250)^{2/3}(950)}{(250+3t)^{2/3}}.$$

Now, we aren't quite done. We want to find $A(t)$ for the t at which the tank is full. Since the amount of liquid in the tank is given by $250 + 3t$, we write

$$850 = 250 + 3t \;\Rightarrow\; 600 = 3t \;\Rightarrow\; t = 200.$$

Thus, the tank is full when $t = 200$. Finally,

$$A(200) = 4(250 + 600) - \frac{(250)^{2/3}(950)}{(250 + 600)^{2/3}} \approx \boxed{2{,}979.85 \text{ lbs}}.$$

2. Additional Examples

EXAMPLES.

1. Suppose the height of a particular plant is given by the function $h(t)$ where t is measured in days. If the plant grows at a rate of $h' = th + t$ inches per day, how long will it take for the plant to grow to 3 feet? (Round your answer to the nearest hundredth.)

Solution: Observe that the height of the plant is measured in inches, which means we are asked to find the t such that $h(t) = 36$. Now,

$$h' = th + t \;\Rightarrow\; h' - th = t$$

is a FOLDE and so we apply our method.

Step 1: Find P, Q

$$P(t) = -t, \quad Q(t) = t$$

Step 2: Find the integrating factor

$$u(t) = e^{\int P(t)\,dt} = e^{\int -t\,dt} = e^{-t^2/2}$$

Step 3: Set up equation (7)

$$\begin{aligned} h \cdot u(t) &= \int Q(t)u(t)\,dt \\ \Rightarrow \quad he^{-t^2/2} &= \int t \cdot e^{-t^2/2}\,dt \\ &= -e^{-t^2/2} + C \\ \Rightarrow \quad h &= -1 + Ce^{t^2/2} \end{aligned}$$

Since we may assume that $h(0) = 0$, we see that

$$0 = -1 + Ce^0 = -1 + C \;\Rightarrow\; C = 1.$$

Thus,

$$h(t) = -1 + e^{t^2/2}.$$

We need to solve $h(t) = 36$ for t:

$$\begin{aligned} 36 &= -1 + e^{t^2/2} \\ \Rightarrow \quad 37 &= e^{t^2/2} \\ \Rightarrow \quad \ln 37 &= t^2/2 \\ \Rightarrow \quad 2\ln 37 &= t^2 \\ \Rightarrow \quad (2\ln 37)^{1/2} &= t. \end{aligned}$$

We conclude it takes $t = (2\ln 37)^{1/2} \approx \boxed{2.69 \text{ days}}$.

2. A 10,000 ft^3 room initially has a radon level of 910 picocuries/ft^3. A ventilation system is installed that brings in 450 ft^3 of air per hour which contains 10 picocuries/ft^3, while an equal quantity of well-mixed air leaves the room each hour. Set up and use a differential equation to determine how long it will take for the room to reach a safe-to-breathe level of 100 picocuries/ft^3. (Round your answer to the nearest hundredth.)

Solution: This is an extremely tricky problem and it takes a lot of patience for both the set up and solution.

First, we should notice that, in principle, this should be very similar to the tank problem because it involves quantities entering, mixing, and leaving a space.

Let $r(t)$ be the amount of radon (in picocuries) per ft^3 per hour (note that this is different than the *total* amount of radon in the room per hour). Our setup should look like

$$r'(t) = [\text{Rate of Radon/ft}^3 \text{ in}] - [\text{Rate of Radon/ft}^3 \text{ out}].$$

This is a little difficult to work with because none of our information is really given in picocuries/ft^3 per hour. So let's write it as

$$\begin{aligned} r'(t) &= \frac{[\text{Rate of Total Radon in}] - [\text{Rate of Total Radon out}]}{\text{Total ft}^3 \text{ in Room}} \\ &= \frac{[\text{Rate of Total Radon in}] - [\text{Rate of Total Radon out}]}{10{,}000}. \end{aligned}$$

$\underline{[\text{Rate of Total Radon in}]}$: There are 10 picocuries of radon entering the room per ft^3 of air per hour, we write this as

$$\left(\frac{10 \text{ picocuries}}{1 \text{ ft}^3}\right)\left(\frac{450 \text{ ft}^3}{1 \text{ hour}}\right) = \frac{4{,}500 \text{ picocuries}}{1 \text{ hour}} = 4{,}500 \text{ picocuries/hour}.$$

This is the *total* amount of radon entering the room per hour.

$\underline{[\text{Rate of Total Radon out}]}$: We are told that 450 ft^3 of well-mixed air leaves the room per hour. This is represented by

$$\left(\frac{\text{total radon in room in picocuries}}{10{,}000 \text{ ft}^3}\right)\left(\frac{450 \text{ ft}^3}{1 \text{ hour}}\right).$$

But observe that

$$r(t) = \frac{\text{total radon in room in picocuries}}{10{,}000 \text{ ft}^3}.$$

Hence, the rate out is just

$$\underbrace{\left(\frac{\text{total radon in room in picocuries}}{10{,}000 \text{ ft}^3}\right)}_{r(t)}\left(\frac{450 \text{ ft}^3}{1 \text{ hour}}\right) = 450r(t) \text{ picocuries/hour}$$

Putting this all together, we have

$$\begin{aligned} r'(t) &= \frac{[\text{Rate of Total Radon in}] - [\text{Rate of Total Radon out}]}{10{,}000} \\ &= \frac{4{,}500 - 450r(t)}{10{,}000} \\ &= \frac{4{,}500}{10{,}000} - \frac{450r(t)}{10{,}000} \\ &= \frac{9}{20} - \frac{9r(t)}{200} \end{aligned}$$

Therefore, our differential equation is

$$r'(t) = \frac{9}{20} - \frac{9r(t)}{200}.$$

But we aren't done yet, because we are asked to find the t such that $r(t) = 100$. So we must solve this differential equation.

This is a FOLDE, but it isn't quite in the correct form yet, so we write

$$r'(t) + \frac{9r(t)}{200} = \frac{9}{20}.$$

Now, we can use our steps to solve it.

Step 1: Find P, Q

$$P(t) = \frac{9}{200}, \quad Q(t) = \frac{9}{20}$$

Step 2: Find the integrating factor

$$\begin{aligned} u(t) &= e^{\int P(t)\,dt} \\ &= e^{\int \frac{9}{200}\,dt} \\ &= e^{9t/200} \end{aligned}$$

Step 3: Set up equation (7)

$$r(t) \cdot u(t) = \int Q(t)u(t)\,dt$$

$$\Rightarrow \quad r(t)\underbrace{e^{9t/200}}_{u(t)} = \int \underbrace{\left(\frac{9}{20}\right)}_{Q(t)} \underbrace{e^{9t/200}}_{u(t)}\,dt$$

$$\Rightarrow \quad r(t)e^{9t/200} = 10e^{9t/200} + C$$

$$\Rightarrow \quad r(t) = 10 + \frac{C}{e^{9t/200}}$$

We were told that $r(0) = 910$. This is to say that

$$\underbrace{910}_{r(0)} = 10 + \frac{C}{e^0} = 10 + C$$

and so we conclude $C = 900$. Hence

$$r(t) = 10 + 900e^{-9t/200}.$$

Finally, we want to find t such that $r(t) = 100$. We write

$$10 + 900e^{-9t/200} = 100$$

$$\Rightarrow \quad 900e^{-9t/200} = 90$$

$$\Rightarrow \quad e^{-9t/200} = \frac{1}{10}$$

$$\Rightarrow \quad \ln(e^{-9t/200}) = \ln\left(\frac{1}{10}\right)$$

$$\Rightarrow \quad -\frac{9}{200}t = \ln\left(\frac{1}{10}\right)$$

$$\Rightarrow \quad t = -\frac{200}{9}\ln\left(\frac{1}{10}\right) \approx \boxed{51.17 \text{ hours}}.$$

3. A corporation is initially worth 6 million dollars and is growing in value. Let V denote the value of the company. Suppose V is growing by 22% each year and is additionally gaining 24% of a growing market estimated at $100e^{.22t}$ million dollars, where t is the number of years the company has existed. Approximate the value of the company after 7 years. (Round to the nearest million dollars.)

<u>**Solution**</u>: Let $V(t)$ be the value of the company in millions of dollars after t years of existence. We are told that V is changing in the following ways:

– it is growing by 22% each year

– it is gaining 24% of a market estimated to be $100e^{.22t}$

Thus, we write

$$\frac{dV}{dt} = \underbrace{.22V}_{\substack{22\% \text{ of} \\ \text{current value}}} + \underbrace{.24\left(100e^{.22t}\right)}_{\substack{24\% \text{ of} \\ \text{emerging market}}} = .22V + 24e^{.22t}$$

This is a FOLDE but is not quite in the correct form. We write

$$\frac{dV}{dt} = .22V + 24e^{.22t} \;\Rightarrow\; \frac{dV}{dt} - .22V = 24e^{.22t}.$$

We go through our steps:

Step 1: Find P, Q

$$P(t) = -.22, \quad Q(t) = 24e^{.22t}$$

Step 2: Find the integrating factor

$$\begin{aligned} u(t) &= e^{\int P(t)\,dt} \\ &= e^{\int(-.22)\,dt} \\ &= e^{-.22t} \end{aligned}$$

Step 3: Set up equation (7)

$$\begin{aligned} V(t)\cdot u(t) &= \int Q(t)u(t)\,dt \\ \Rightarrow \quad V(t)\underbrace{\left(e^{-.22t}\right)}_{u(t)} &= \int \underbrace{\left(24e^{.22t}\right)}_{Q(t)}\underbrace{\left(e^{-.22t}\right)}_{u(t)}\,dt \\ &= \int 24e^{.22t-.22t}\,dt \\ &= \int 24e^{0}\,dt \\ &= \int 24\,dt \\ V(t)e^{-.22t} &= 24t + C \\ \Rightarrow \quad V(t) &= \frac{24t+C}{e^{-.22t}} \\ &= e^{.22t}(24t+C) \end{aligned}$$

We are given that $V(0) = 6$, and so

$$6 = \underbrace{e^{.22\cdot 0}}_{1}(24\underset{\substack{\uparrow \\ 0}}{(0)} + C) = C.$$

Hence,

$$V(t) = e^{.22t}(24t+6).$$

Therefore, after 7 years, the company is worth approximately

$$V(7) = e^{.22 \cdot 7}(24 \cdot 7 + 6) \approx \boxed{812 \text{ million dollars}}.$$

4. Find the integrating factor of

$$(\sin 2x)y' - 2(\cot 2x)y = -\cos 2x, \quad 0 < x < \frac{\pi}{4}.$$

Solution: The $0 < x < \frac{\pi}{4}$ is only meant to indicated where a valid solution exists as a function. We do not explicitly use this information.

First, recall that $\cot 2x = \frac{\cos 2x}{\sin 2x}$. So we are really looking at

$$(\sin 2x)y' - 2\left(\frac{\cos 2x}{\sin 2x}\right) y = -\cos 2x.$$

Second, this is a FOLDE but it isn't in quite the correct form. Divide both sides by $\sin 2x$ to get

$$y' - 2\left(\frac{\cos(2x)}{(\sin(2x))(\sin(2x))}\right) y = -\frac{\cos(2x)}{\sin(2x)}$$

$$\Rightarrow \quad y' - 2\left(\frac{\cos(2x)}{\sin^2(2x)}\right) y = -\frac{\cos(2x)}{\sin(2x)}$$

In this form, it is clear that

$$P(x) = -2\left(\frac{\cos(2x)}{\sin^2(2x)}\right).$$

Because our goal is to find the integrating factor, we need only determine $P(x)$. The integrating factor $u(x)$ is given by the formula

$$u(x) = e^{\int P(x)\,dx}.$$

So we will need to find

$$\int P(x)\,dx = \int -2\left(\frac{\cos(2x)}{\sin^2(2x)}\right) dx.$$

This is a substitution problem, but we don't want to use u since we are using $u(x)$ to mean the integrating factor.

Let $t = \sin(2x)$, then $dt = 2\cos(2x)\,dx$, which means

$$\begin{aligned}\int -2\left(\frac{\cos(2x)}{\sin^2(2x)}\right) dx &= \int -\frac{1}{t^2}\,du \\ &= \frac{1}{t} \\ &= \frac{1}{\sin(2x)} \\ &= \csc(2x)\end{aligned}$$

Therefore, our integrating factor is

$$u(x) = e^{\int P(x)\,dx} = \boxed{e^{\csc 2x}}.$$

5. Find the general solution to

$$(x-4)y' + y = x^2 + 4.$$

Solution: This is not in the correct form for us to apply equation (7). We divide everything by $x-4$ to get

$$y' + \frac{1}{x-4}y = \frac{x^2+4}{x-4}$$

Next, we go through our steps.

Step 1: Find P, Q

$$P(x) = \frac{1}{x-4}, \quad Q(x) = \frac{x^2+4}{x-4}$$

Step 2: Find the integrating factor

$$\begin{aligned} u(x) &= e^{\int P(x)\,dx} \\ &= e^{\int 1/(x-4)\,dx} \\ &= e^{\ln|x-4|} \\ &= |x-4| = x-4 \end{aligned}$$

where we assume that $x > 4$.

Step 3: Set up equation (7)

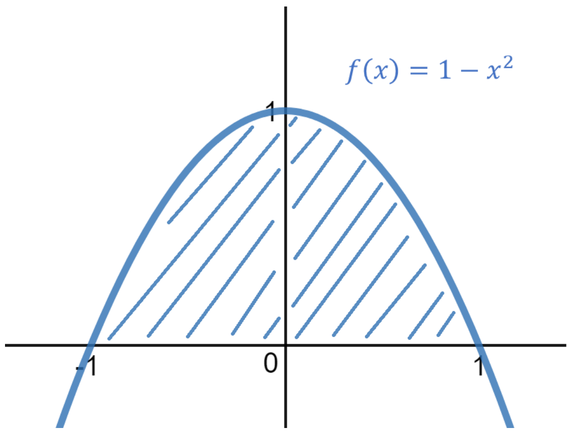
$$\begin{aligned} & y \cdot u(x) = \int Q(x)u(x)\,dx \\ \Rightarrow \quad & y\underbrace{(x-4)}_{u(x)} = \int \underbrace{\left(\frac{x^2+4}{x-4}\right)}_{Q(x)} \underbrace{(x-4)}_{u(x)}\,dx \\ \Rightarrow \quad & (x-4)y = \int (x^2+4)\,dx \\ \Rightarrow \quad & (x-4)y = \frac{1}{3}x^3 + 4x + C \\ \Rightarrow \quad & y = \boxed{\frac{\frac{1}{3}x^3 + 4x + C}{x-4}} \end{aligned}$$

Lesson 11: Area Between Two Curves

1. Area between Two Curves

We know how to find the area under a curve. Today we determine how to find the area of a region bounded by two or more curves.

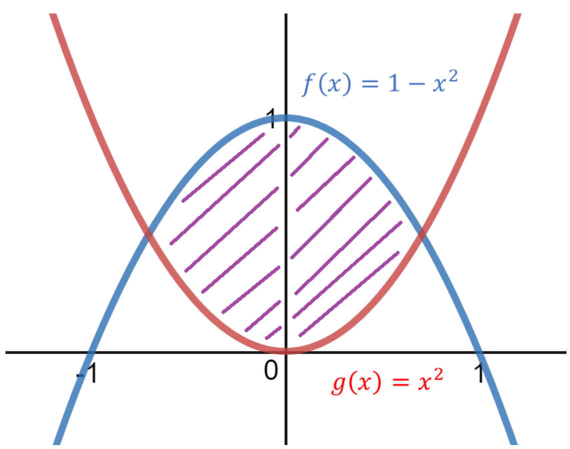
Ex 1. We know how to find the area underneath $f(x) = 1 - x^2$.



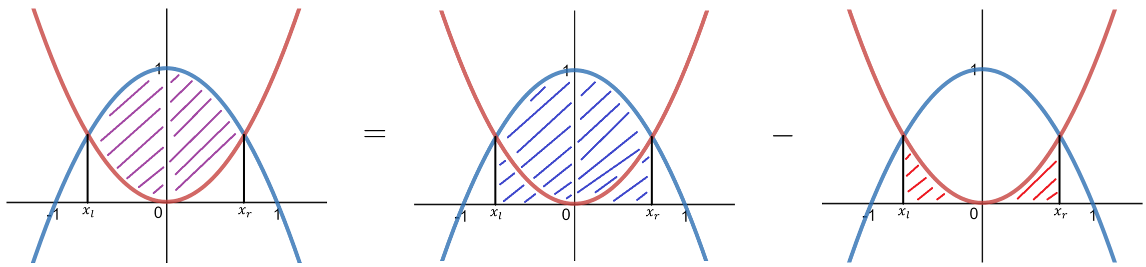
This is just the integral

$$\int_{-1}^{1} (1 - x^2)\, dx.$$

But what if we wanted to find the area between $f(x) = 1 - x^2$ and $g(x) = x^2$? Consider the following graph:



Taking a moment to think about it, it should be clear that if we take the area under the curve $f(x) = 1 - x^2$ and subtract the area under the curve $g(x) = x^2$, then we should get the purple area. We can represent this visually as follows:



Although this gives the right idea visually, it isn't clear from the picture exactly what the area is as a number. To find the area between curves algebraically, we need two things: (1) where the functions intersect (if they do at all) and (2) which function is "larger".

(1) To determine where the functions intersect, set them equal to each other:

$$
\begin{aligned}
x^2 &= 1 - x^2 \\
\Rightarrow \quad 2x^2 &= 1 \\
\Rightarrow \quad x^2 &= \frac{1}{2} \\
\Rightarrow \quad x &= \pm\frac{1}{\sqrt{2}}.
\end{aligned}
$$

Thus, our functions intersect at $\pm\frac{1}{\sqrt{2}}$. Next, we need to determine which is function is larger on the interval $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

(2) To determine which function is larger, we need only check a **single point** between $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ (think about why this is true). For convenience, we can check $x = 0$. Plugging in $x = 0$, the function $1 - x^2$ is clearly larger than x^2.

From this information, we can setup our integral:

$$
\underbrace{\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (1 - x^2)\, dx}_{\uparrow \text{ larger function}} - \underbrace{\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x^2\, dx}_{\uparrow \text{ smaller function}} = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} [1 - x^2 - x^2]\, dx.
$$

Check on your own that this integral is equal to $\frac{2\sqrt{2}}{3}$.

In general, the area between $f(x)$ and $g(x)$ is given by

$$
\int_{x_L}^{x_R} f(x)\, dx - \int_{x_L}^{x_R} g(x)\, dx = \int_{x_L}^{x_R} [f(x) - g(x)]\, dx
$$

where $f(x) > g(x)$ on the interval $x_L \le x \le x_R$.

NOTE 29. We need to choose the correct "larger" function, else our area will be negative.

EX 2. We can also find the area between two functions of y. Consider the functions $F(y) = 8 - y^2$ and $G(y) = y^2$. Since these are functions of y rather than of x, these are harder to graph. So we should go about this algebraically.

We apply the same two steps as above:

(1) We set our functions ($8 - y^2$ and y^2) equal to each other to see where they intersect.

$$\begin{aligned} y^2 &= 8 - y^2 \\ \Rightarrow \quad 2y^2 &= 8 \\ \Rightarrow \quad y^2 &= 4 \\ \Rightarrow \quad y &= \pm 2 \end{aligned}$$

(2) We check which function is larger on the interval $[-2, 2]$. Taking the convenient point $y = 0$, we see that $8 - y^2$ is larger on this interval (note that since functions of y give x-values, the larger function is the one to the **right**).

Our integral is then

$$\int_{-2}^{2} (8 - y^2)\, dy - \int_{-2}^{2} y^2\, dy = \int_{-2}^{2} (8 - y^2 - y^2)\, dy.$$

NOTE 30. When setting up integrals, consider the following:

(a) Functions of x have x-values for bounds and are integrated with respect to x

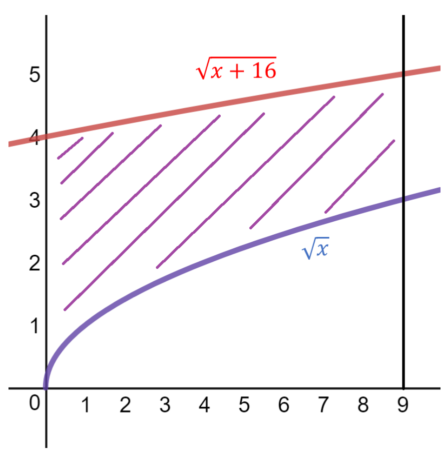
(b) Functions of y have y-values for bounds and are integrated with respect to y

EXAMPLES.

1. Find the area between the curves

$$y = \sqrt{x}, \quad y = \sqrt{x + 16}, \quad 0 \le x \le 9.$$

Solution: Here, we are given the bounds over which to integrate, so we need only determine which function is larger on that interval. Consider the following graph:



By the graph, we see that $\sqrt{x+16}$ is the larger function. Therefore, the area between the curves is given by

$$\int_0^9 \left[\sqrt{x+16} - \sqrt{x}\right]\, dx = \left(\frac{2}{3}\right)(x+16)^{3/2} - \left(\frac{2}{3}\right)(x)^{3/2}\Big|_0^9$$

$$= \left(\frac{2}{3}(9+16)^{3/2} - \frac{2}{3}(9)^{3/2}\right) - \left[\frac{2}{3}(16)^{3/2} - \frac{2}{3}(0)^{3/2}\right]$$

$$= \frac{2}{3}(25)^{3/2} - \frac{2}{3}(27) - \frac{2}{3}(64)$$

$$= \frac{2}{3}[125 - 27 - 64]$$

$$= \frac{2}{3}(34)$$

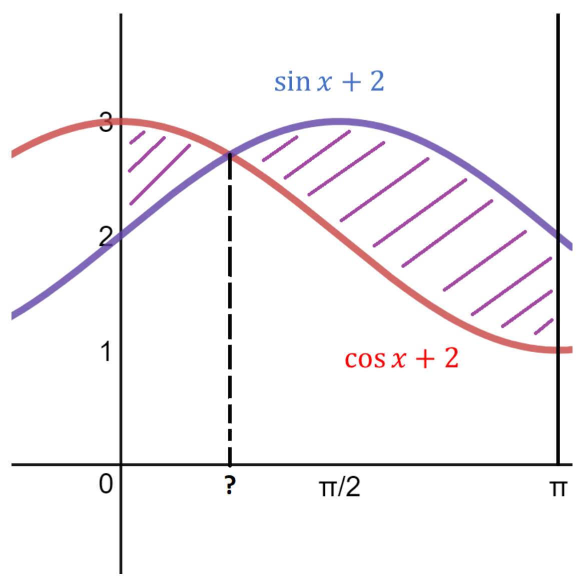
$$= \boxed{\frac{68}{3}}$$

Remark 31. Observe that if we went about this algebraically, we would have written $\sqrt{x} = \sqrt{x+16}$ which has **no** solutions. What does this mean? It means the functions never intersect and thus one function is **always** larger than the other function.

2. Find the area between

$$y = \sin x + 2, \quad y = \cos x + 2, \quad 0 \le x \le \pi.$$

Solution: The problem has given us the interval over which we should integrate, so we may focus *only* on x-values in the interval $[0, \pi]$. For this problem, sketching the picture gives the most intuition about what is happening although we will ultimately need to use algebra to set up the integral. Consider the following graph:



We see that on one part of the interval, $\cos x+2$ is larger than $\sin x+2$, but not on the rest of the interval. We need to determine where these functions intersect (so that we know the point at which the functions switch positions). Write

$$\cos x + 2 = \sin x + 2 \quad \Rightarrow \quad \cos x = \sin x$$

Now, if we think about the unit circle, $\cos x = \sin x$ for $0 \le x \le \pi$ when $x = \frac{\pi}{4}$. If we had not graphed this function, this extra point of intersection (in addition to the interval they gave us) would imply that we need to split the integral into two different intervals and, on each interval, check which function is larger.

Quickly checking which function is larger where (try the points $x = 0$ and $x = \frac{\pi}{2}$), our area is given by

$$\begin{aligned}
&\int_0^{\pi/4} [(\cos x + 2) - (\sin x + 2)]\, dx + \int_{\pi/4}^{\pi} [(\sin x + 2) - (\cos x + 2)]\, dx \\
&= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx \\
&= \sin x + \cos x \Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi} \\
&= \left[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)\right] + \left[(-\cos\pi - \sin\pi) - \left(\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)\right] \\
&= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - 1\right] + \left[1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] \\
&= \frac{2\sqrt{2}}{2} - 1 + 1 + \frac{2\sqrt{2}}{2} \\
&= \frac{4\sqrt{2}}{2} = \boxed{2\sqrt{2}}
\end{aligned}$$

3. Find the area bounded by

$$x = y^2 - y, \quad x + y = 1.$$

Solution: We quickly rewrite the given information as

$$x = y^2 - y, \quad x = 1 - y.$$

Here, we are not given the bounds over which we should integrate, which means we need to integrate between where the functions intersect. To find where they intersect, write

$$\begin{aligned}
& y^2 - y = 1 - y \\
\Rightarrow\quad & y^2 = 1 \\
\Rightarrow\quad & y = \pm 1
\end{aligned}$$

Thus, we are integrating from -1 to 1. We need to see which function is larger than the other in this interval. Check $y = 0$:

$$x(0) = 0^2 - 0 = 0$$

$$x(0) = 1 - 0 = 1$$

Thus, we see $x = 1 - y$ is larger than $x = y^2 - y$ between $y = -1$ and $y = 1$. So the area is given by

$$\begin{aligned}
\int_{-1}^{1} [(1-y) - (y^2 - y)]\, dy &= \int_{-1}^{1} [1 - y^2]\, dy \\
&= y - \frac{1}{3}y^3 \Big|_{-1}^{1} \\
&= 1 - \frac{1}{3}(1)^3 - \left[(-1) - \frac{1}{3}(-1)^3\right] \\
&= 1 - \frac{1}{3} - \left[-1 + \frac{1}{3}\right] \\
&= 2 - \frac{2}{3} \\
&= \boxed{\frac{4}{3}}
\end{aligned}$$

2. Additional Examples

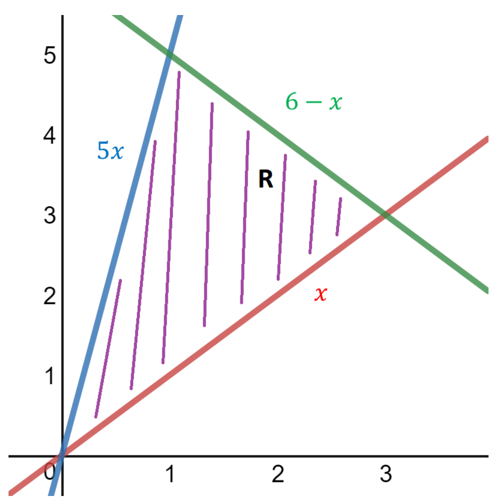
EXAMPLES.

1. Find the equation of the vertical line that divides the area of the region R bounded by

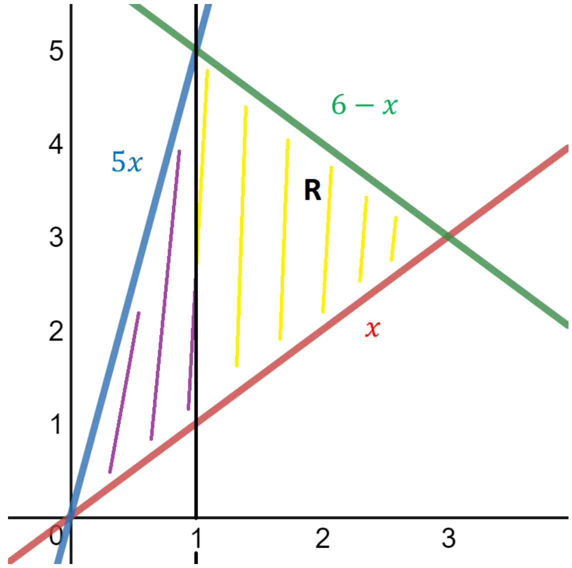
$$y = x, \quad y = 5x, \quad y = 6 - x$$

in half.

Solution: Recall that a vertical line is given by the equation $x = a$ where a is some number. To start this problem, we should sketch a graph:



Before we can determine which vertical line divides the of R in half, we need to find the area of the R. Observe that the area in this triangle is given in two parts: the region trapped between $y = x$ and $y = 5x$ and the region trapped between $y = x$ and $y = 6 - x$.



To compute the area of R, we first need to compute the area of the purple region, then the area of the yellow region, and add these values together. Observe we are integrating with respect to x, which means our bounds will be x-values.

Purple Region: This is the region between $y = x$ and $y = 5x$. These functions clearly intersect at $x = 0$. Next, we observe that

$$\begin{aligned} 5x &= 6 - x \\ \Rightarrow \quad 6x &= 6 \\ \Rightarrow \quad x &= 1 \end{aligned}$$

This is pertinent because when $x = 1$, we are no longer describing the purple region but rather the yellow region.

The area of the purple region is given by

$$\begin{aligned} \int_0^1 [5x - x]\,dx &= \int_0^1 4x\,dx \\ &= 2x^2\Big|_0^1 \\ &= 2(1)^2 - 2(0)^2 = 2 \end{aligned}$$

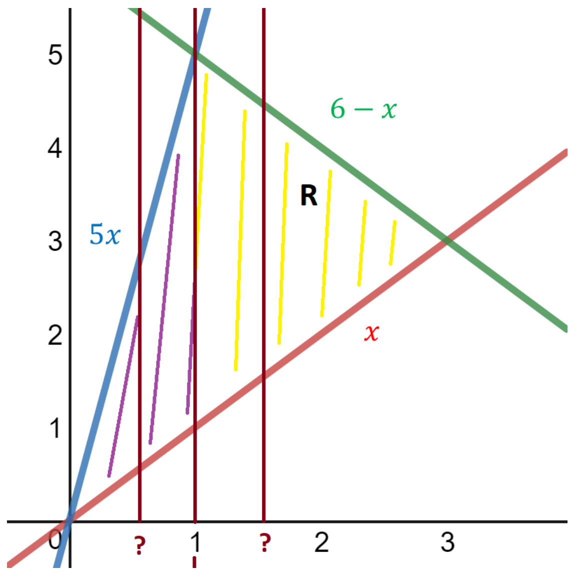
Yellow Region: The two curves describing this region are $y = x$ and $y = 6 - x$. We see that they intersect when $x = 3$. Therefore, the area of the yellow region is given by

$$\int_1^3 [6 - x - x]\,dx = \int_1^3 [6 - 2x]\,dx$$

$$= 6x - x^2 \Big|_1^3$$
$$= 6(3) - 3^2 - [6(1) - 1^2]$$
$$= 18 - 9 - 6 + 1 = 4$$

We conclude the area of R is $2 + 4 = 6$.

We now want to determine for what value a does the vertical line $x = a$ divide R in half. We have 3 options:



Written in words, we mean

(1) $x = a$ could be in the purple region (then $0 \leq a < 1$)

(2) $x = a$ could be the line that divides R into the purple and yellow regions (then $a = 1$)

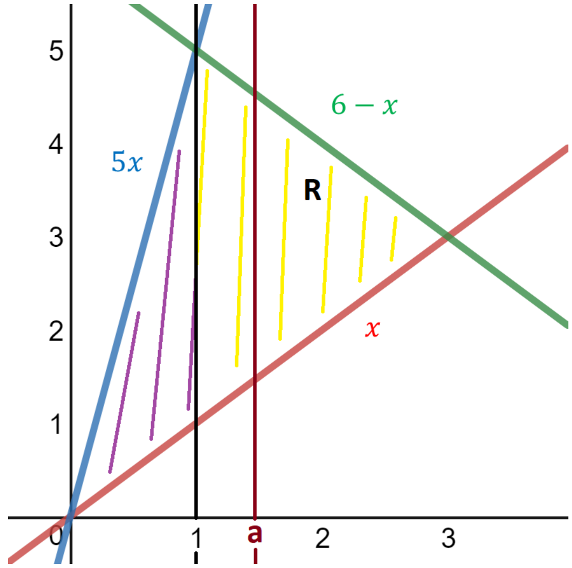
(3) $x = a$ could be in the yellow region (then $1 < a \leq 3$)

Let's take a moment to consider: if $x = a$ is in the purple region or if $x = 1$, then the area on its left is *at most* 2 because the entire area of the purple region is 2. But half of the area of R is 3. So we **must** have $a > 1$. Thus, $x = a$ is in the **yellow region**.

Now that we know what region $x = a$ lies in, we can put together the following equation:

$$\int_0^1 4x \, dx + \int_1^a [6 - 2x] \, dx = 3. \tag{8}$$

What does this mean? The RHS is exactly half of the area of R (because the area of R is 6). The LHS include the area of the purple region and the area of the yellow region **up to the line** $x = a$:



Now, we know that

$$\underbrace{\int_0^1 4x\,dx}_{\substack{\text{area of}\\ \text{purple region}}} = 2.$$

So, our equation (8) becomes

$$2 + \int_1^a [6-2x]\,dx = 3 \quad \Rightarrow \quad \int_1^a [6-2x]\,dx = 1.$$

Integrating, we have

$$\begin{aligned}\int_1^a [6-2x]\,dx &= 6x - x^2\Big|_1^a \\ &= 6(a) - a^2 - [6(1) - (1)^2] \\ &= 6a - a^2 - 6 + 1 \\ &= 6a - a^2 - 5 \\ &= -a^2 + 6a - 5.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\int_1^a [6-2x]\,dx = 1 \\ \Rightarrow \quad & -a^2 + 6a - 5 = 1 \\ \Rightarrow \quad & -a^2 + 6a - 6 = 0\end{aligned}$$

Applying the quadratic formula, we get two potential values for a:

$$a = 3 - \sqrt{3} \quad \text{or} \quad a = 3 + \sqrt{3}.$$

Because $x = a$ is in the yellow region, we must have $1 < a \leq 3$. Thus we choose $a = 3 - \sqrt{3}$ and conclude the vertical line which cuts region R in half is

$$\boxed{x = 3 - \sqrt{3}}.$$

NOTE 32. You could have also approached this by focusing on the right side of R and writing

$$\int_a^3 [6 - 2x]\, dx = 3.$$

2. A company reports that profits for the last fiscal year were 14.2 million dollars. Given that t is the number of years from now, the company predicts that profits will grow for the next 7 years at a continuous annual rate between 3.6% and 5.8%. Estimate the cumulative difference in predictive total profits over the 7 years based on the predictive range of growth rates. Round to 3 decimal places.

Solution: The key to this question is recalling that continuous annual growth is given by

$$A(t) = Pe^{rt}$$

where P is the invested amount, r is the rate, and t is time in years. Here, we are investing 14.2 million dollars. Because we want the cumulative difference, we are really asked to find the difference over $0 \leq t \leq 7$ between the functions

$$f(x) = 14.2e^{.036t} \quad \text{and} \quad g(x) = 14.2e^{.058t}.$$

Observe that $g(x)$ is larger than $f(x)$ for all values between $0 \leq t \leq 7$. Hence, we need only compute

$$\int_0^7 [14.2e^{.058t} - 14.2e^{.036t}]\, dt.$$

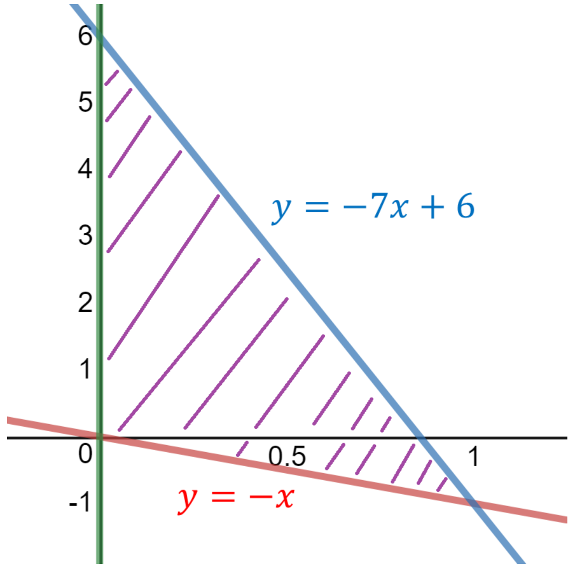
We write

$$\begin{aligned}
&\int_0^7 14.2[e^{.058t} - e^{.036t}]\, dt \\
&= 14.2 \left(\frac{1}{.058} e^{.058t} - \frac{1}{.036} e^{.036t} \right) \Bigg|_0^7 \\
&= 14.2 \left[\left(\frac{1}{.058} e^{.058 \cdot 7} - \frac{1}{.036} e^{.036 \cdot 7} \right) - \left(\frac{1}{.058} e^{.058 \cdot 0} - \frac{1}{.036} e^{.036 \cdot 0} \right) \right] \\
&\approx \boxed{9.564 \text{ million dollars}}
\end{aligned}$$

3. Find the area of the triangular region bounded by the y-axis and the curves

$$y = -x \quad \text{and} \quad y = -7x + 6.$$

Solution: We begin by sketching a picture of the region:



We see that $y = -7x+6$ is the larger function, so we need only determine the bounds. One bound is clearly $x = 0$. To find the other, we set $y = -7x+6$ equal to $y = -x$.

$$\begin{aligned} & -7x + 6 = -x \\ \Rightarrow \quad & -6x + 6 = 0 \\ \Rightarrow \quad & -6x = -6 \\ \Rightarrow \quad & x = 1 \end{aligned}$$

Thus, the area is given by

$$\begin{aligned} \int_0^1 [-7x + 6 - (-x)]\, dx &= \int_0^1 [-6x + 6]\, dx \\ &= -3x^2 + 6x \Big|_0^1 \\ &= -3(1)^2 + 6(1) - \left[-3(0)^2 + 6(0)\right] \\ &= -3 + 6 \\ &= \boxed{3} \end{aligned}$$

4. Find the area of the region bounded by

$$y = 5x^4 - 5x^2 \quad \text{and} \quad y = 10x^2$$

to the right of the y-axis.

Solution: We determine where the functions intersect and then algebraically determine which is larger for $x \geq 0$ (since we area only consider the region to the right of the y-axis). Setting these functions equal to each other:

$$\begin{aligned} & 5x^4 - 5x^2 = 10x^2 \\ \Rightarrow \quad & 5x^4 - 15x^2 = 0 \end{aligned}$$

$$\Rightarrow \quad 5x^2(x^2-3)=0$$

which implies that either $x=0$ or $x=\pm\sqrt{3}$. Again, we only take $x \geq 0$, so our bounds are $x=0$ and $x=\sqrt{3}$.

Next, we determine which function is larger on $[0,\sqrt{3}]$. Since $\sqrt{3}>1$, we check the point $x=1$.

$$5(1)^4-5(1)^2=5-5=0$$

$$10(1)^2=10$$

Hence, we conclude that $y=10x^2$ is the larger function on this interval.

The area of this region is therefore

$$\begin{aligned}
\int_0^{\sqrt{3}} \left[10x^2-\left(5x^4-5x^2\right)\right]\,dx &= \int_0^{\sqrt{3}} [15x^2-5x^4]\,dx \\
&= \frac{15}{3}x^3-\frac{5}{5}x^5\Big|_0^{\sqrt{3}} \\
&= 5x^3-x^5\Big|_0^{\sqrt{3}} \\
&= 5(\sqrt{3})^3-(\sqrt{3})^5-[5(0)^3-(0)^5] \\
&= 5(3)\sqrt{3}-9\sqrt{3} \\
&= 15\sqrt{3}-9\sqrt{3} \\
&= \boxed{6\sqrt{3}}
\end{aligned}$$

Lesson 12: Volume of Solids of Revolution (I)

1. Disk Method

Solids of revolution are 3-dimensional shapes that come from regions in the xy-plane revolved about a line. Spheres, cones, and cylinders are all examples of solids of revolution. A particularly interesting aspect of solids of revolution is that we can compute their volume. In fact, using the techniques learned today, we can derive the formula for volume of a generic sphere or cone (although this particular application is not explored in this class).

Ex 1. Consider the function $f(x) = \sin x$ [on left] on the interval from 0 to π revolved about the x-axis [on right]:

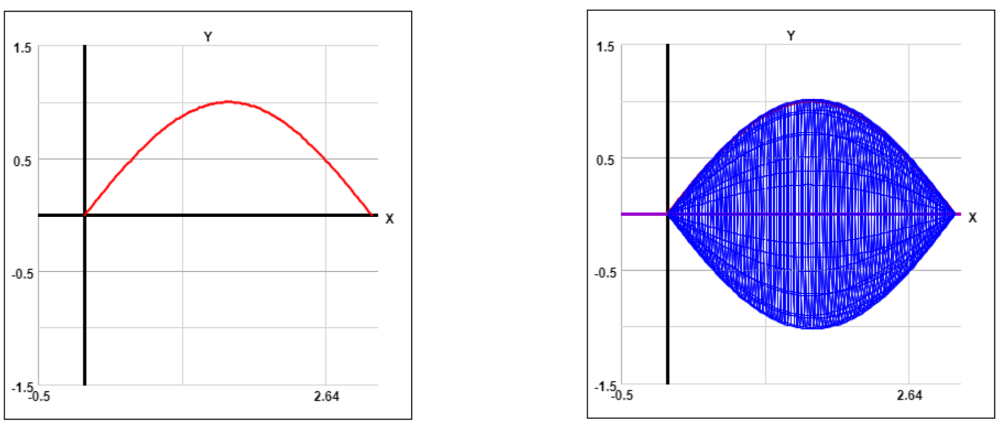


FIGURE 3. $f(x) = \sin(x)$ for $0 \leq x \leq \pi$ revolved about the x-axis

Our solid of revolution on the right is a sort of lemon shape but we do not have an explicit formula that gives the volume. Our goal is to find a method of determining volume for these types of objects.

The idea is this: slice the solid into very thin disks. The volume of any disk is given by $\pi r^2 w$ where r is the radius and w is the width of the disk. Because we are slicing our object very thinly, the width of each disk will be dx (which is something we always think of as very small). Once we find the volume of all these thin disks, we add them up using an integral.

The technical details of this type of problem boils down to determining the radius of each disk, which is the difference of the function and the line of revolution. For this example, the radius of each disk is $\sin x$ because, if we are revolving about the x-axis, the radius of each disk is just the height of the function. If we know the radius

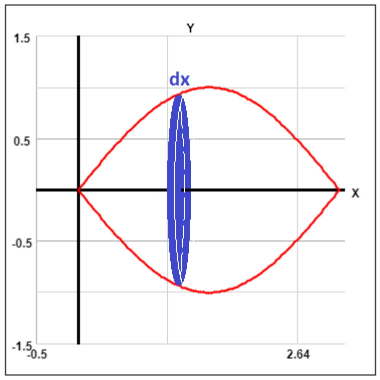


FIGURE 4. An example thin disk.

is $\sin x$ and the width is dx, our formula for disks gives

$$\text{Volume} = \int_0^{\pi} \pi(\underset{\substack{\uparrow \\ \text{radius}}}{\sin x})^2 \underset{\substack{\uparrow \\ \text{width}}}{dx} .$$

This method of finding the volume is called the **disk method** and is given by the formula

$$\text{Volume} = \int_{x_L}^{x_R} \pi(f(x))^2\, dx$$

where x_L is the bound to the left, x_R is the bound to the right, and $f(x)$ is the function we are revolving about the x-axis.

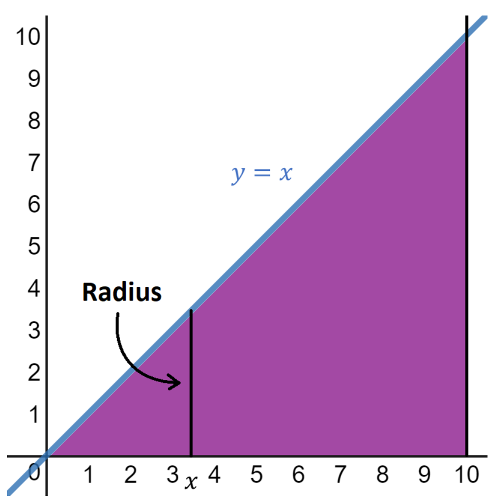
EXAMPLES.

1. Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y = x,\ x = 0,\ x = 10,\ \text{ and } y = 0$$

about the x-axis.

Solution: If we sketch a picture, this becomes quite easy.



The lower bound is $x = 0$, the upper bound is $x = 10$, and the function is $f(x) = x$, so the volume is given by

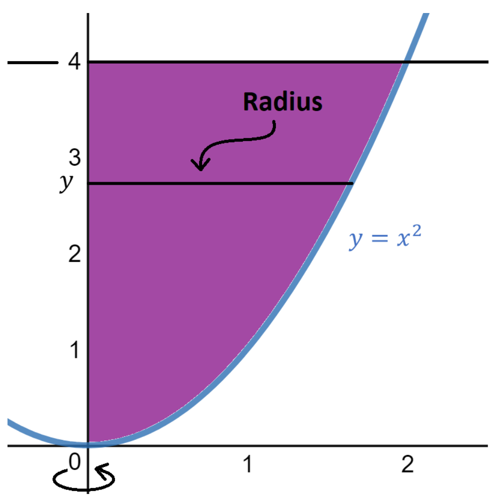
$$\begin{aligned}\text{Volume} &= \pi \int_0^{10} (\underset{\substack{\uparrow\\ \text{radius}}}{x})^2 \underset{\substack{\uparrow\\ \text{width}}}{dx}\\ &= \pi \int_0^{10} x^2\, dx\\ &= \frac{\pi}{3} x^3 \Big|_0^{10}\\ &= \boxed{\frac{\pi}{3}(1000)}\end{aligned}$$

2. Find the volume of the solid generated by revolving the given region in the first quadrant about the y-axis:

$$y = x^2,\ x = 0,\ \text{and } y = 4.$$

Solution: There is a big difference between this problem and Example 1. Here, we are revolving about the y-axis, **not** the x-axis. Be sure to read each question carefully so you know where you are revolving the region.

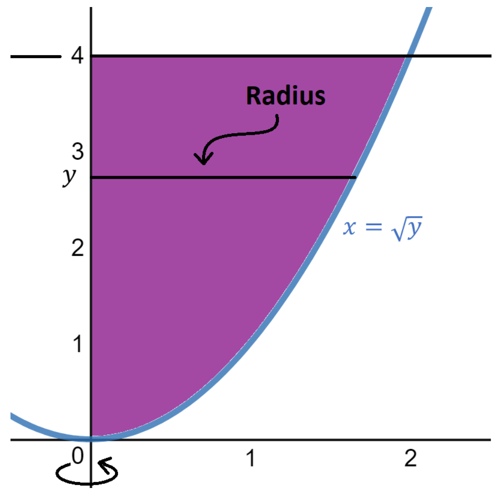
We sketch a picture of the graph:



Because we are revolving about the y-axis, the radius is **not** x^2. In this situation, we are integrating with respect to y, which means the radius is actually a function of y. How do we determine this function? Well,

$$y = x^2 \ \Rightarrow\ x = \sqrt{y}.$$

Our sketch doesn't change but the labeling does:



Now, we write

$$\begin{aligned}\text{Volume} &= \int_0^4 \pi\Big(\underset{\substack{\uparrow\\ \text{radius}}}{\sqrt{y}} \Big)^2 \underset{\substack{\uparrow\\ \text{width}}}{dy} \\ &= \pi \int_0^4 (\sqrt{y})^2\, dy \\ &= \pi \int_0^4 y\, dy \\ &= \frac{\pi}{2} y^2 \Big|_0^4 \\ &= \frac{\pi}{2}(4)^2 \\ &= \boxed{8\pi}\end{aligned}$$

The disk method about the y-axis is given by

$$\text{Volume} = \int_{y_B}^{y^T} \pi (g(y))^2\, dy$$

where y_B is the bottom bound, y^T is the top bound, and $g(y)$ is the function we are revolving about the y-axis.

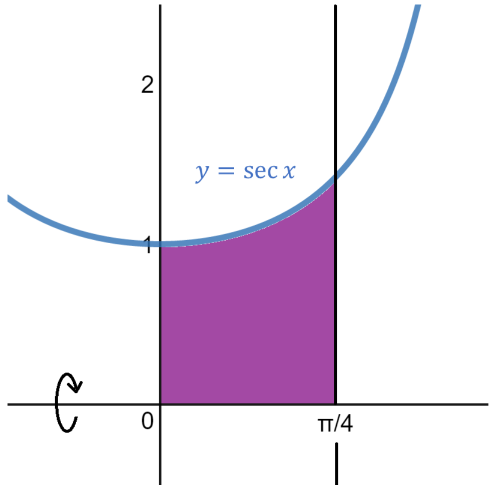
NOTE 33. If we are revolving around the x-axis, the radius is $y = f(x)$. If we are revolving around the y-axis, the radius is $x = g(y)$. But just as before with integration: functions of x have bounds in x and functions of y have bounds in y.

3. Find the volume of the solid generated by revolving the region enclosed by the curves

$$y = \sec x,\ y = 0,\ x = 0,\ x = \frac{\pi}{4}$$

about the x-axis.

Solution: This graph is tougher to sketch:



The radius is $\sec x$ and we have bounds $x = 0, x = \frac{\pi}{4}$. We write

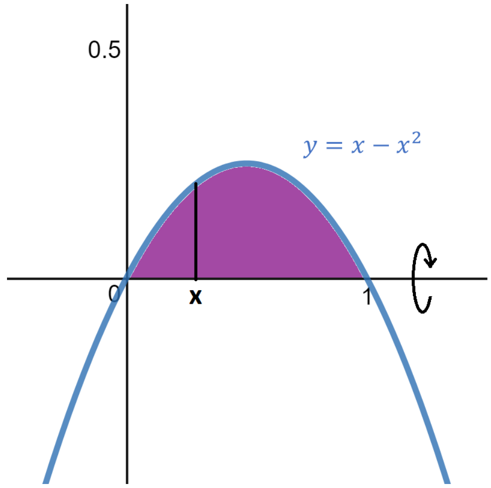
$$\begin{aligned}
\text{Volume} &= \int_0^{\pi/4} \pi(\sec x)^2\, dx \\
&= \pi \int_0^{\pi/4} \sec^2 x\, dx \\
&= \pi \tan x \Big|_0^{\pi/4} \\
&= \pi \left[\tan \frac{\pi}{4} - \tan 0\right] \\
&= \pi[1 - 0] \\
&= \boxed{\pi}
\end{aligned}$$

4. Find the volume of the solid obtained by revolving the region bounded by $y = x - x^2$ and the x-axis about the x-axis.

Solution: We want to be able to sketch these types of functions quickly so here are some things to notice about $y = x - x^2$: we can rewrite it as $x - x^2 = x(1 - x)$ which means it has roots at $x = 0, x = 1$. Further, the coefficient of x^2 is negative, so the parabola will open down. With all this information, we are able to sketch the graph. However, we don't *need* to draw a picture in this case since the volume is just given by the integral

$$\text{Volume} = \int_0^1 \pi(x - x^2)^2\, dx.$$

That being stated, for certain problems, we need intuition for how the graph looks and so taking time to practice drawing graphs for simpler cases is important. This will be especially true for the next two sections.



Computing the volume of the solid, we write

$$\begin{aligned}
\text{Volume} &= \int_0^1 \pi(x - x^2)^2\,dx \\
&= \pi \int_0^1 (x - x^2)(x - x^2)\,dx \\
&= \pi \int_0^1 (x^2 - 2x^3 + x^4)\,dx \\
&= \pi \left(\frac{1}{3}x^3 - \frac{2}{4}x^4 + \frac{1}{5}x^5 \right) \Big|_0^1 \\
&= \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) \\
&= \boxed{\frac{\pi}{30}}
\end{aligned}$$

5. Find the volume of the solid that results by revolving the region bounded by the curves

$$y = 3\sqrt{1 - x^2},\ y = 0,\ \text{and } x = 0$$

about the y-axis.

Solution: We are told that $y = 3\sqrt{1 - x^2}$ but we are revolving about the y-axis, which means we want to solve for x:

$$\begin{aligned}
y &= 3\sqrt{1 - x^2} \\
\Rightarrow \quad \frac{1}{3}y &= \sqrt{1 - x^2} \\
\Rightarrow \quad \frac{1}{9}y^2 &= 1 - x^2 \\
\Rightarrow \quad x^2 &= 1 - \frac{1}{9}y^2
\end{aligned}$$

$$\Rightarrow \quad x = \sqrt{1 - \frac{1}{9}y^2}$$

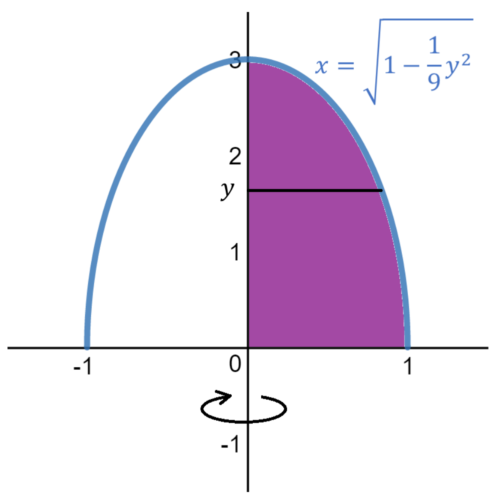
Note that formulas of the type

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are **ellipses**. Ellipses always pass through the points

$$(a, 0), \quad (-a, 0), \quad (0, b), \quad (0, -b).$$

As it happens, our equation is an ellipse with $a = 1$ and $b = 3$. Now, since we are given the curves $y = 0$ and $x = 0$ also bound the region, we assume we are only talking about the region in the first quadrant. Therefore, our picture is going to look like



Finally, we write

$$\begin{aligned}
\text{Vol} &= \pi \int_0^3 \left(\sqrt{1 - \frac{1}{9}y^2} \right)^2 \, dy \\
&= \pi \int_0^3 \left(1 - \frac{1}{9}y^2 \right) \, dy \\
&= \pi \left(y - \frac{1}{27}y^3 \right) \Big|_0^3 \\
&= \pi \left(3 - \frac{3^3}{27} \right) \\
&= \pi \, (3 - 1) \\
&= \boxed{2\pi}
\end{aligned}$$

2. Additional Examples

EXAMPLES.

1. Find the volume of the solid generated by revolving the region enclosed by the curves

$$y = \frac{4}{x}, \; y = 0, \; x = 9, \; x = 15$$

about the x-axis.

Solution: The radius is given by $y = \dfrac{4}{x}$ and so our volume is

$$\begin{aligned}
\text{Volume} &= \int_9^{15} \pi \left(\frac{4}{x}\right)^2 \, dx \\
&= \int_9^{15} \frac{16\pi}{x^2} \, dx \\
&= \int_9^{15} 16\pi x^{-2} \, dx \\
&= 16\pi \left(\frac{1}{-2+1}\right) x^{-2+1} \Big|_9^{15} \\
&= 16\pi \left(\frac{1}{-1}\right) x^{-1} \Big|_9^{15} \\
&= -\frac{16\pi}{x} \Big|_9^{15} \\
&= -\frac{16\pi}{15} - \left[-\frac{16\pi}{9}\right] \\
&= \frac{16\pi}{9} - \frac{16\pi}{15} \\
&= \boxed{\frac{32\pi}{48}}
\end{aligned}$$

2. Find the volume of the solid generated by revolving the region enclosed by the curves

$$x + y = \frac{21}{8}, \; x = 0, \; y = 0$$

about the y-axis.

Solution: Since we are revolving about the y-axis, our function needs to be a function of y. Hence, our radius is given by

$$x + y = \frac{21}{8} \;\Rightarrow\; x = \frac{21}{8} - y.$$

But we need bounds. The region described by the curves is a triangle where clearly the lowest y bounds is $y = 0$. The upper y bound is where $x = \frac{21}{8} - y$ intersects $x = 0$ (which is just the y-axis). To find this y-value, we write

$$\begin{aligned} 0 &= \frac{21}{8} - y \\ \Rightarrow \quad y &= \frac{21}{8} \end{aligned}$$

Finally, we set up and compute our volume formula:

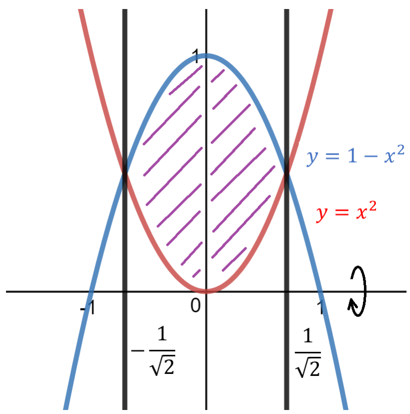
$$\begin{aligned} \text{Volume} &= \int_0^{21/8} \pi \left(\frac{21}{8} - y\right)^2 dy \\ &= \int_0^{21/8} \pi \left(\frac{21}{8} - y\right)\left(\frac{21}{8} - y\right) dy \\ &= \int_0^{21/8} \pi \left(\left(\frac{21}{8}\right)^2 - 2\left(\frac{21}{8}\right) y + y^2\right) dy \\ &= \int_0^{21/8} \pi \left(\frac{441}{64} - \frac{21}{4} y + y^2\right) dy \\ &= \pi \left[\frac{441}{64} y - \frac{21}{8} y^2 + \frac{1}{3} y^3\right]_0^{21/8} \\ &= \pi \left[\frac{441}{64}\left(\frac{21}{8}\right) - \frac{21}{8}\left(\frac{21}{8}\right)^2 + \frac{1}{3}\left(\frac{21}{8}\right)^3 - \left[\frac{441}{64}(0) - \frac{21}{8}(0)^2 + \frac{1}{3}(0)^3\right]\right] \\ &= \pi \left[\frac{9261}{512} - \frac{9261}{512} + \frac{1}{3}\left(\frac{9261}{512}\right)\right] \\ &= \boxed{\frac{3087\pi}{512}} \end{aligned}$$

Lesson 13: Volume of Solids of Revolution (II)

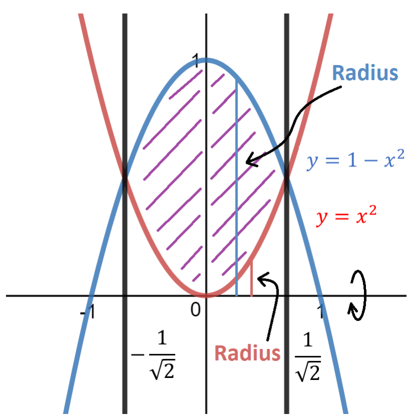
1. Washer Method

We continue discussing finding the volume of solids of revolution. We introduce the **washer method**. This method is used whenever there is a "gap" between **what** we are revolving and **where** we are revolving.

Ex 1. Suppose we have two functions $f(x) = 1 - x^2$ and $g(x) = x^2$ and we want to revolve the area **between** them about the x-axis.



We need to take the volume of the disks obtained from revolving $f(x) = 1 - x^2$ about the x-axis and *subtract* the volume of the disks obtained from revolving $g(x) = x^2$ about the x-axis:



The volume of a disk with radius $1 - x^2$ is

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \pi(1 - x^2)^2 \, dx$$

and the volume of a disk with radius x^2 is

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \pi(x^2)^2\, dx.$$

Thus, the volume of this solid of revolution is given by

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \pi(1-x^2)^2\, dx - \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \pi(x^2)^2\, dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \pi\left[(1-x^2)^2 - (x^2)^2\right]\, dx.$$

NOTE 34. This is **not** the same as

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \pi(1-x^2-x^2)^2\, dx$$

because

$$(a+b)^2 \neq a^2+b^2.$$

To see this, observe that

$$(2+3)^2 = 5^2 = 25 \neq 13 = 4+9 = 2^2+3^2.$$

Setting $(a+b)^2 = a^2+b^2$ is called the **Freshman's Dream** and it is **incorrect**.

The washer method about the x-axis is given by

$$\text{(9)} \qquad \text{Volume} = \int_{x_L}^{x_R} \pi\left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right]\, dx.$$

In Ex 1 above, we see the outer radius was $1-x^2$ and the inner radius was x^2.

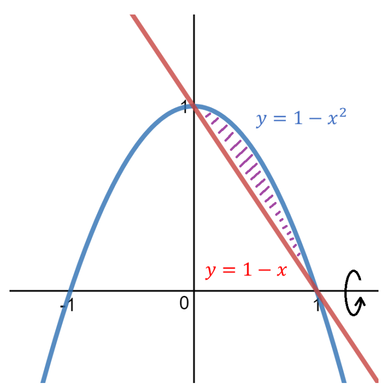
The **outer radius** is the function *further* from where we are revolving and the **inner radius** is the function *closer* to where we are revolving. Think of the outer radius as creating the solid and the inner radius as the part we need to remove.

EXAMPLES.

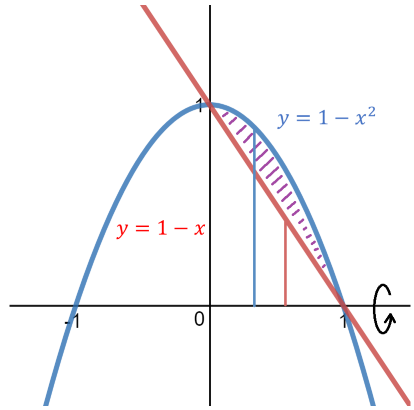
1. Find the volume of the solid obtained by revolving the given region about the x-axis:

$$y = 1-x^2, \quad y = 1-x.$$

Solution: We sketch a quick graph of this region:



Our bounds for this region will be $0 \leq x \leq 1$. Next, we determine the outer radius and inner radius. Consider

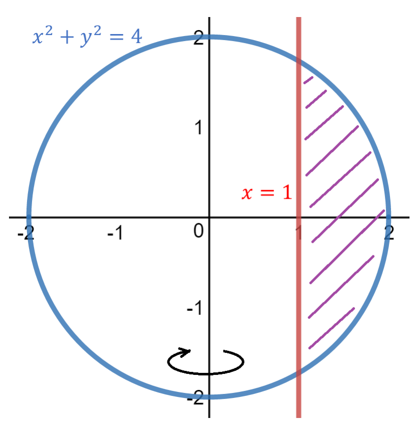


Hence, the outer radius is $1 - x^2$ and the inner radius is $1 - x$. Therefore, by equation (9),

$$\begin{aligned}
\text{Volume} &= \int_0^1 \pi \left[(1 - x^2)^2 - (1 - x)^2\right] dx \\
&= \pi \int_0^1 \left[(x^4 - 2x^2 + 1) - (1 - 2x + x^2)\right] dx \\
&= \pi \int_0^1 (x^4 - 3x^2 + 2x)\, dx \\
&= \pi \left(\frac{1}{5}x^5 - \frac{3}{3}x^3 + \frac{2}{2}x^2\right)\Bigg|_0^1 \\
&= \pi \left(\frac{1}{5} - 1 + 1\right) \\
&= \boxed{\frac{\pi}{5}}
\end{aligned}$$

2. Find the volume of the solid generated by revolving the region inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$ about the y-axis.

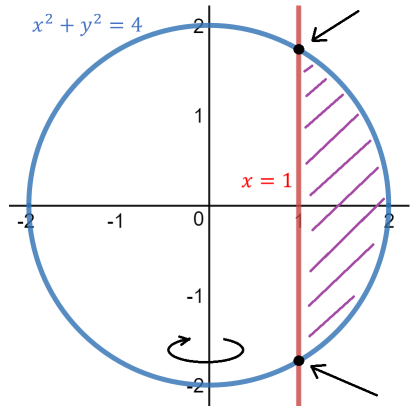
Solution: The region should look like:



To derive this picture, observe that $x^2+y^2=4$ describes a circle of radius 2 centered at the origin.

NOTE 35. $x^2+y^2=r^2$ is a circle of radius r centered at the origin.

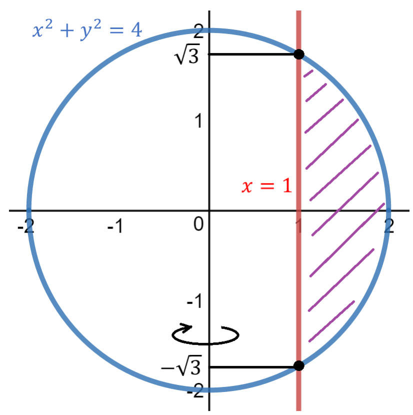
Since we are revolving around the y-axis, our bounds must be in terms of y. So we need to find the y-values where $x^2+y^2=4$ intersects $x=1$.



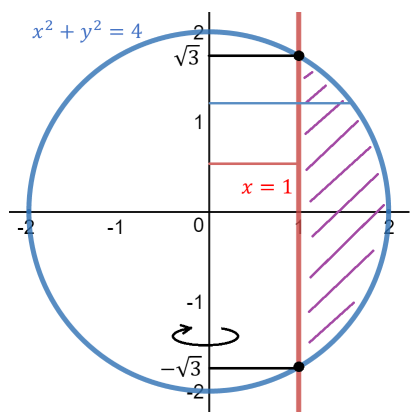
This circle intersects the line $x=1$, so substituting this into $x^2+y^2=4$,

$$(1)^2+y^2=4 \;\Rightarrow\; y^2=3.$$

This means our bounds will be $-\sqrt{3}\le y\le\sqrt{3}$.



Next, we need to find our outer radius and inner radius. If we look at our sketch,



then we see that inner radius is $x = 1$ and we note that the outer radius will be a function of y since we are revolving about the y-axis.

Our function is $x^2 + y^2 = 4$ and we need to solve for x (because then we would have a function of y). Write

$$\begin{aligned} &x^2 + y^2 = 4 \\ \Rightarrow \quad &x^2 = 4 - y^2 \\ \Rightarrow \quad &x = \pm\sqrt{4 - y^2}. \end{aligned}$$

So we have a choice here, we can take either

$$x = \sqrt{4 - y^2} \quad \text{or} \quad x = -\sqrt{4 - y^2}.$$

Well, we want only $x \geq 1$, so we take $x = \sqrt{4 - y^2}$ as our outer radius.

Finally, we can set up our volume function:

$$\text{Volume} = \int_{-\sqrt{3}}^{\sqrt{3}} \pi \left[\left(\sqrt{4 - y^2} \right)^2 - (1)^2 \right] dx.$$

Our final step is to solve this integral. We have

$$\begin{aligned} \text{Volume} &= \int_{-\sqrt{3}}^{\sqrt{3}} \pi \left[\left(\sqrt{4 - y^2} \right)^2 - (1)^2 \right] dy \\ &= \pi \int_{-\sqrt{3}}^{\sqrt{3}} (4 - y^2 - 1)\, dy \\ &= \pi \int_{-\sqrt{3}}^{\sqrt{3}} (3 - y^2)\, dy \\ &= \pi \left[3y - \frac{1}{3} y^3 \right]_{-\sqrt{3}}^{\sqrt{3}} \\ &= \pi \left[\left(3(\sqrt{3}) - \frac{1}{3}(\sqrt{3})^3 \right) - \left(3(-\sqrt{3}) - \frac{1}{3}(-\sqrt{3})^3 \right) \right] \\ &= \pi \left[6\sqrt{3} - \frac{2}{3}(\sqrt{3})^3 \right] \\ &= \pi \left[6\sqrt{3} - \frac{2}{3}(3\sqrt{3}) \right] \\ &= \pi \left[6\sqrt{3} - 2\sqrt{3} \right] \\ &= \boxed{4\sqrt{3}\pi} \end{aligned}$$

NOTE 36. When we are rotating about the y-axis, the washer method is

$$\text{(10)} \qquad \text{Volume} = \int_{y_B}^{y_T} \pi \left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2 \right] \, dy.$$

3. Find the volume of the solid that results from revolving the region enclosed by

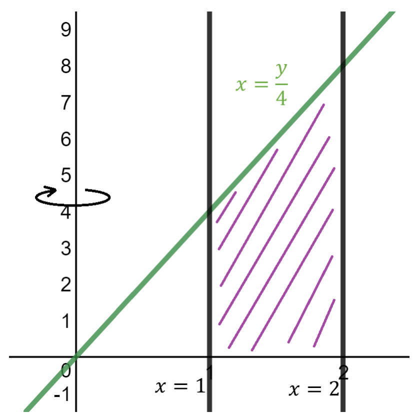
$$y = 4x, \quad x = 1, \quad x = 2, \quad y = 0$$

about the y-axis. Round your answer to the nearest hundredth.

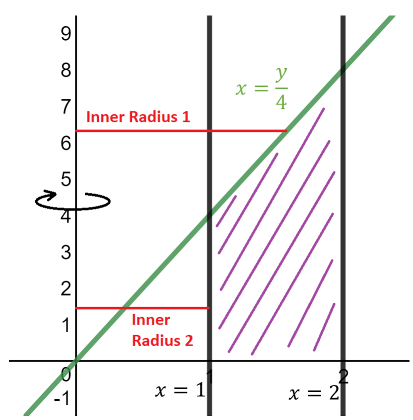
Solution: Firstly, since we are revolving around the y-axis, we need to rewrite our function so that it gives us x-values. But this is easy:

$$y = 4x \;\Rightarrow\; \frac{y}{4} = x.$$

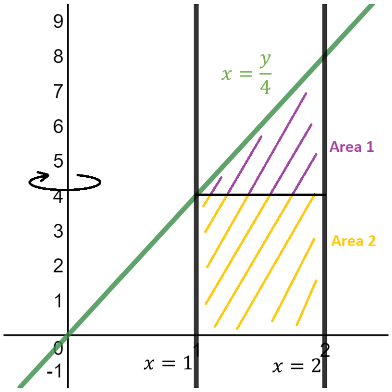
Now, we want to sketch a graph. Although this region is very simple, if you don't draw the picture correctly, you are going to miss an important detail.



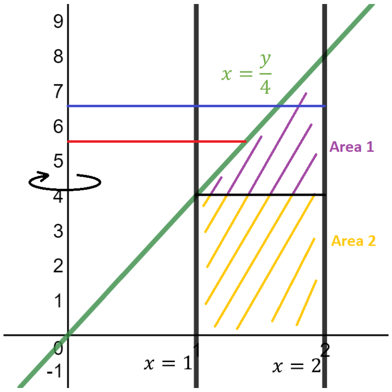
We now need to find the outer radius and inner radius. Observe that depending on the y-values, the inner radius is given by **different** functions. This is difficult to detect algebraically. For problems about solids of revolution, it is best to sketch a picture (this will require taking time to practice sketching functions). Not drawing a quick graph may lead to missing key features of the solid.



To address the issue of different inner radii, we break the region into two areas such that each area has only one function for its inner radius:



For Area 1,



we have

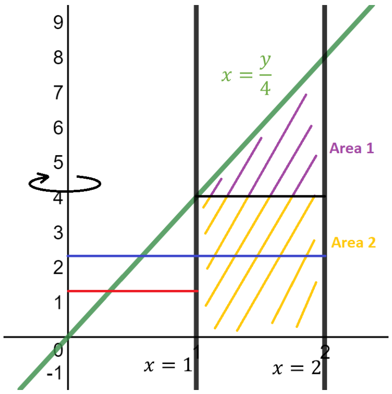
$$\text{Outer Radius}: \ x = 2$$

$$\text{Inner Radius}: \ x = \frac{y}{4}$$

$$\text{Bounds}: \ 4 \leq y \leq 8$$

Observe that these bounds are in terms of y because we are revolving about the y-axis.

For Area 2,



we have

$$\text{Outer Radius}: \ x = 2$$

$$\text{Inner Radius}: \ x = 1$$

$$\text{Bounds}: \ 0 \leq y \leq 4$$

Thus, we get our formula for volume:

$$\begin{aligned}
\text{Volume} &= \underbrace{\int_0^4 \pi \left[(2)^2 - (1)^2\right] \, dy}_{\text{Area 2}} + \underbrace{\int_4^8 \pi \left[(2)^2 - \left(\frac{y}{4}\right)^2\right] \, dy}_{\text{Area 1}} \\
&= \pi \int_0^4 3 \, dy + \pi \int_4^8 \left(4 - \frac{y^2}{16}\right) \, dy \\
&= \pi \left[\int_0^4 3 \, dy + \int_4^8 \left(4 - \frac{y^2}{16}\right) \, dy\right] \\
&= \pi \left[3y \Big|_0^4 + \left(4y - \frac{1}{48}y^3\right) \Big|_4^8\right] \\
&= \pi \left[(3(4) - 3(0)) + \left[4(8) - \frac{1}{48}(8)^3 - \left(4(4) - \frac{1}{48}(4)^3\right)\right]\right] \\
&= \pi \left[12 + 32 - \frac{512}{48} - 16 + \frac{64}{48}\right] \\
&= \pi \left[12 + 16 - \frac{448}{48}\right] \\
&= \pi \left[\frac{56}{3}\right] \\
&\approx \boxed{58.64}
\end{aligned}$$

2. Additional Examples

EXAMPLES.

1. The equation $\frac{x^2}{9} + \frac{y^2}{16} = 1$ describes an ellipse. Find the volume of the solid generated by this region being revolved around

   (a) the x-axis

   (b) the y-axis

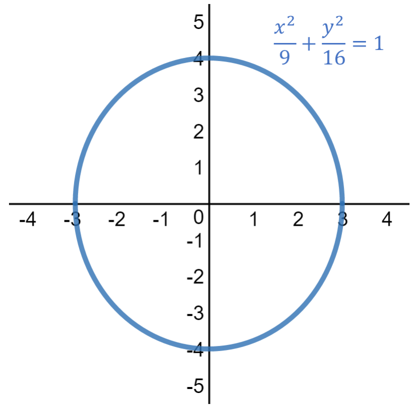
NOTE 37. An ellipse centered at the origin is always given by something of the form $\frac{x^2}{a} + \frac{y^2}{b} = 1$. Further, this ellipse will go through the points

$$(\sqrt{a}, 0), \quad (-\sqrt{a}, 0), \quad (0, \sqrt{b}), \quad (0, -\sqrt{b}).$$

In this case, this means our ellipse will pass through the points

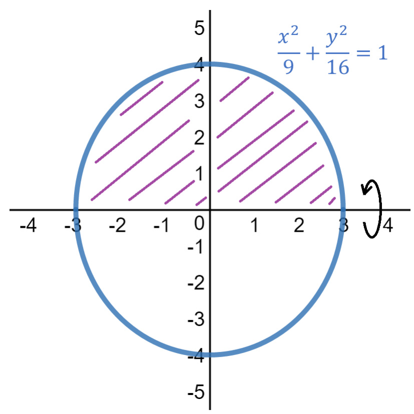
$$(3, 0), (-3, 0), (0, 4), (0, -4).$$

So, our picture should look like



Solution:

(a) We want to revolve about the x-axis, which means we should focus on this region:



Here, we don't need to use the washer method because the inner radius is just 0. We do, however, need to determine the outer radius. Because we are revolving about the x-axis, we need a function of x. So, we take $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ and solve for y. Write

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$\Rightarrow \quad \frac{y^2}{16} = 1 - \frac{x^2}{9}$$

$$\Rightarrow \quad y^2 = 16 - \frac{16}{9}x^2$$

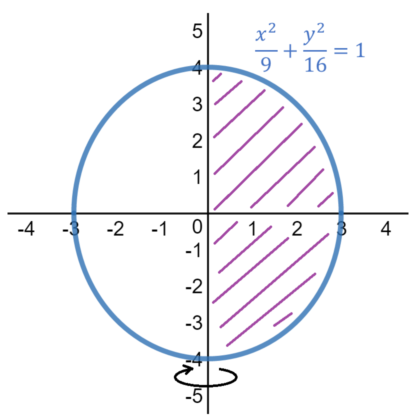
$$\Rightarrow \quad y = \sqrt{16 - \frac{16}{9}x^2}$$

where we take the positive y-values because we are *above* the x-axis.

We observe that our bounds are $-3 \le x \le 3$. So, our volume is given by

$$\begin{aligned}
\text{Volume} &= \int_{-3}^{3} \pi \left(\sqrt{16 - \frac{16}{9}x^2}\right)^2 dx \\
&= \int_{-3}^{3} \pi \left(16 - \frac{16}{9}x^2\right) dx \\
&= \pi \left[16x - \frac{16}{27}x^3\right]_{-3}^{3} \\
&= \pi \left[\left(16(3) - \frac{16}{27}(3)^3\right) - \left(16(-3) - \frac{16}{27}(-3)^3\right)\right] \\
&= \pi \left[48 - 16 - (-48 + 16)\right] \\
&= \pi[32 - (-32)] = \boxed{64\pi}
\end{aligned}$$

(b) Because we are revolving around the y-axis, we need to focus on this region:



Again, we can just use the disk method because we only have one radius. We are revolving around the y-axis, which means we need a function of y. Write

$$\begin{aligned}
&\frac{x^2}{9} + \frac{y^2}{16} = 1 \\
\Rightarrow \quad &\frac{x^2}{9} = 1 - \frac{y^2}{16} \\
\Rightarrow \quad &x^2 = 9 - \frac{9}{16}y^2
\end{aligned}$$

$$\Rightarrow \quad x = \sqrt{9 - \frac{9}{16}y^2}$$

where we take the positive x-values because we are to the *right* of the y-axis.

Our bounds here are $-4 \le y \le 4$, which means our volume is given by

$$\begin{aligned}
\text{Volume} &= \int_{-4}^{4} \pi \left(\sqrt{9 - \frac{9}{16}y^2}\right)^2 dy \\
&= \int_{-4}^{4} \pi \left(9 - \frac{9}{16}y^2\right) dy \\
&= \pi \left[9y - \frac{9}{16}\left(\frac{1}{3}\right)y^3\right]_{-4}^{4} \\
&= \pi \left[9y - \frac{3}{16}y^3\right]_{-4}^{4} \\
&= \pi \left[\left(9(4) - \frac{3}{16}(4)^3\right) - \left(9(-4) - \frac{3}{16}(-4)^3\right)\right] \\
&= \pi \left[\left(36 - \frac{3}{16}(64)\right) - \left(-36 + \frac{3}{16}(64)\right)\right] \\
&= \pi \left[36 - 12 + 36 - 12\right] = \boxed{48\pi}
\end{aligned}$$

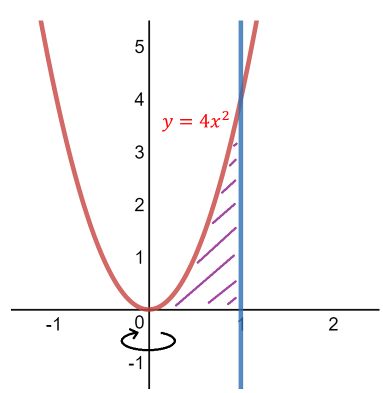
REMARK 38. The values are for (a) and (b) **not** the equal even though they both concern the same ellipse. It's a very special situation when revolving a region around the x- and y-axis gives identical volume in both cases. Be careful to always know where you are asked to revolve.

2. Find the volume of the solid generated by revolving the region by the curves

$$y = 4x^2, \quad x = 1, \quad y = 0$$

about the y-axis.

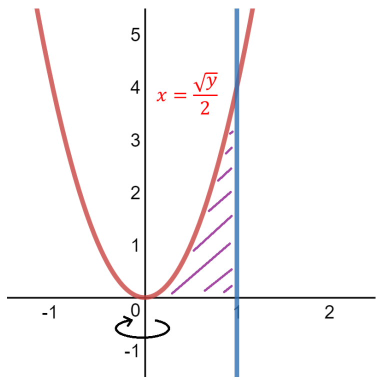
Solution: We should sketch a picture:



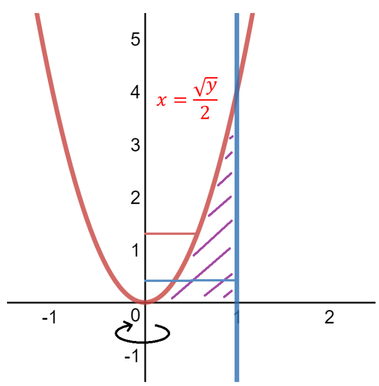
By the picture, it is clear that this will be an application of the washer method. Since we are revolving about the y-axis, we need to rewrite $y = 4x^2$ so that y is the independent variable. Write

$$\begin{aligned} y &= 4x^2 \\ \Rightarrow \quad \frac{y}{4} &= x^2 \\ \Rightarrow \quad \frac{\sqrt{y}}{2} &= x \end{aligned}$$

Relabeling our picture, we have



Now, because we are revolving about the y-axis, our outer radius is $x = 1$ and our inner radius is $x = \dfrac{\sqrt{y}}{2}$:



We need to find the bounds of our integral. We are looking for y-values:

$$\begin{aligned} 1 &= \frac{\sqrt{y}}{2} \\ \Rightarrow \quad 2 &= \sqrt{y} \\ \Rightarrow \quad 4 &= y \end{aligned}$$

Hence, our integral is

$$\text{Volume} = \pi \int_0^4 \left[(1)^2 - \left(\frac{\sqrt{y}}{2} \right)^2 \right] \, dy$$

$$= \pi \int_0^4 \left[1 - \frac{y}{4}\right] \, dy$$

$$= \pi \left[y - \frac{y^2}{8}\right]_0^4$$

$$= \pi \left[4 - \frac{(4)^2}{8} - \left(0 - \frac{0^2}{4}\right)\right]$$

$$= \pi \left[4 - 2\right]$$

$$= \boxed{2\pi}$$

3. The shape of a fuel tank for the wing of a jet aircraft is designed by revolving the region bounded by the function

$$y = \frac{10}{7} x^2 \sqrt{5 - x}$$

and the x-axis, where $0 \le x \le 5$, about the x-axis. Given x and y are in meters, find the volume of the fuel tank. Round your answer to 2 decimal places.

Solution: Since this is about the x-axis, we don't need to make any changes to the function. We write

$$V = \int_0^5 \pi \left(\frac{10}{7} x^2 \sqrt{5 - x}\right)^2 \, dx$$

$$= \int_0^5 \frac{100\pi}{49} x^4 (5 - x) \, dx$$

$$= \int_0^5 \frac{100\pi}{49} (5x^4 - x^5) \, dx$$

$$= \frac{100\pi}{49} \left[x^5 - \frac{1}{6} x^6\right]_0^5$$

$$= \frac{100\pi}{49} \left[5^5 - \frac{1}{6}(5)^6 - \left(0^5 - \frac{1}{6}(0)^6\right)\right]$$

$$\approx \boxed{3339.28 \text{ meters}}$$

Lesson 14: Volume of Solids of Revolution (III)

1. Revolving about Horizontal and Vertical Lines

We continue to expand our understanding of solids of revolution. The key takeaway from today's lesson is that finding the volume of a solid of revolution is all about determining the radii, regardless of where we are revolving our region. Specifically, we will be revolving around horizontal and vertical lines that are not the x- and y-axes. Here, we need to think geometrically about how we determine the radii of our disks. Although this may appear to be a daunting task (revolving about lines that are not the axes), the idea is essentially the same: the radius is the *difference* between the function we are revolving and where we are revolving. The biggest difficulty is matching x-values with x-values and y-values with y-values.

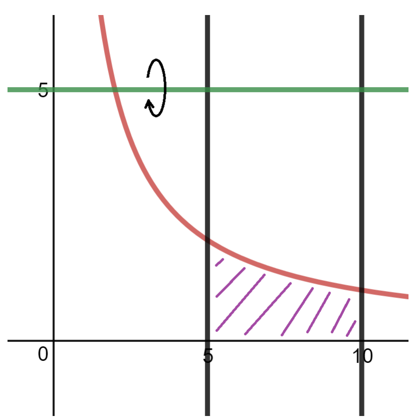
EXAMPLES.

1. Consider the region bounded by the curves

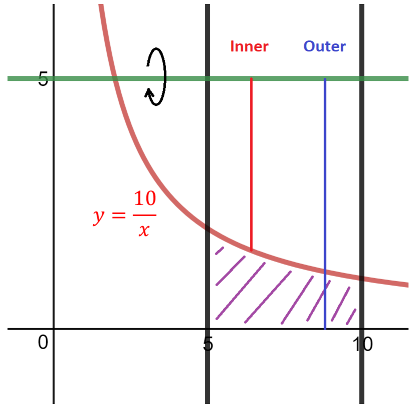
$$y = \frac{10}{x}, \quad y = 0, \quad x = 5, \quad x = 10.$$

(a) Find the volume of the solid generated by revolving the region about the line $y = 5$.

Solution: We sketch a graph to get a geometric understanding of what is going on.



We are **not** revolving around the x-axis, but instead the line $y = 5$ although this is still a *horizontal* line. We need to be careful with how we choose our outer radius and inner radius because now it is relative to $y = 5$ and not the x-axis.



Our outer radius is the difference between $y = 5$ and $y = 0$ and the inner radius is the difference between $y = 5$ and $y = \dfrac{10}{x}$ (think about how you would draw the disks and determine their radii). Then we use exactly the same formula as we have been using for the washer method:

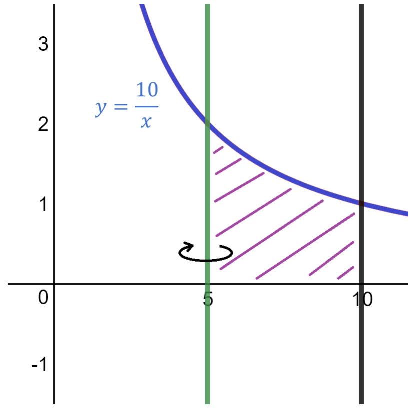
$$\text{Volume} = \int_5^{10} \pi \left[(\underset{\substack{\uparrow \\ \text{outer}}}{5 - 0})^2 - \left(\underset{\substack{\uparrow \\ \text{inner}}}{5 - \frac{10}{x}} \right)^2 \right] dx.$$

The *only* difference between this and the previous methods is determining the radii, every other aspect remains the same. So,

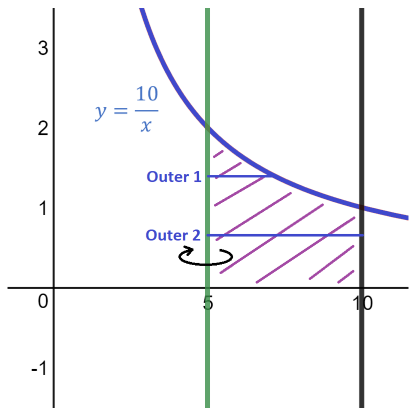
$$\begin{aligned}
\text{Volume} &= \int_5^{10} \pi \left[25 - \left(25 - \frac{100}{x} + \frac{100}{x^2} \right) \right] dx \\
&= \pi \int_5^{10} \left[\frac{100}{x} - \frac{100}{x^2} \right] dx \\
&= \pi \left[100 \ln x + \frac{100}{x} \right]_5^{10} \\
&= \pi \left[\left(100 \ln 10 + \frac{100}{10} \right) - \left(100 \ln 5 + \frac{100}{5} \right) \right] \\
&= \pi \left[100 \ln 10 + 10 - 100 \ln 5 - 20 \right] \\
&= \pi \left[100(\ln 10 - \ln 5) + 10 - 20 \right] \\
&= \boxed{\pi[100 \ln 2 - 10]}
\end{aligned}$$

(b) Find the volume of the solid generated by revolving the region about the line $x = 5$.

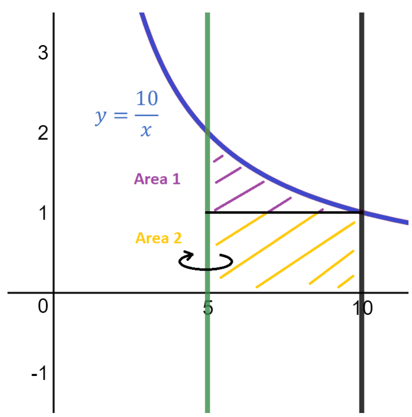
Solution: We go about this with very much the same spirit as in part (a). However, there is a major difference because of where we are revolving. Observe that this is a *vertical* line. Consider the graph



Additionally, we observe that, depending on the y-value, we have different outer radii.



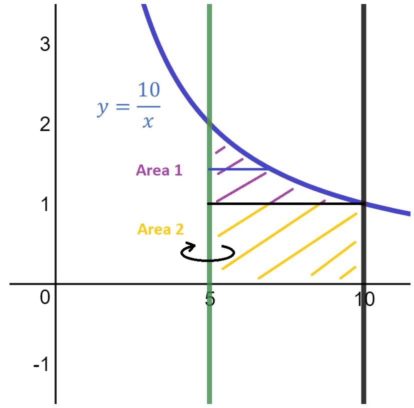
So we need to break the graph into Area 1 and Area 2 based on the two different outer radii functions.



We do want to observe that for either Area 1 or Area 2, the inner radius is 0 because there's no "gap" between where we are revolving and what we are revolving. So we may use the disk method for both areas.

Since we are revolving around the y-axis, we need to solve for x (see note (40)). Given $y = \dfrac{10}{x}$, we get $x = \dfrac{10}{y}$.

For Area 1,

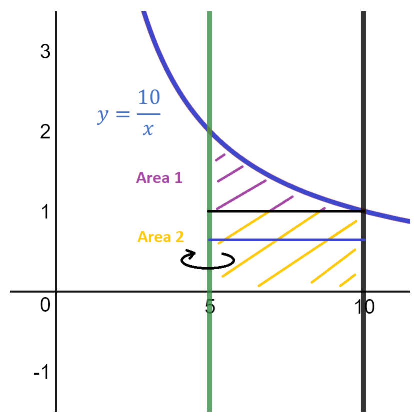


We emphasize that we think of x as a function of y and so the bounds are y-values. This follows whenever we revolve about a vertical line. Hence, we get

$$\text{Radius} : x = \frac{10}{y} - 5$$

$$\text{Bounds} : 1 \leq y \leq 2$$

For Area 2,



we get

$$\text{Radius} : x = 10 - 5$$

$$\text{Bounds} : 0 \leq y \leq 1$$

Remark 39. These radii are the **differences** between the functions and the line we are revolving about. When in doubt, take the function you are revolving and where you are revolving and subtract them. The squaring will take care of any issues with the sign.

Therefore, our volume is given by

$$\text{Volume} = \underbrace{\int_0^1 \pi(10-5)^2 \, dy}_{\text{Area 2}} + \underbrace{\int_1^2 \pi \left(\frac{10}{y} - 5\right)^2 \, dy}_{\text{Area 1}}$$

$$= \int_0^1 \pi(5)^2\,dy + \int_1^2 \pi\left(\frac{10}{y} - 5\right)^2\,dy$$

$$= \pi\left[\int_0^1 25\,dy + \int_1^2 \left(\frac{100}{y^2} - \frac{100}{y} + 25\right)\,dy\right]$$

$$= \pi\left[25y\Big|_0^1 + \left(-\frac{100}{y} - 100\ln y + 25y\right)\Big|_1^2\right]$$

$$= \pi\left[(25-0) + \left(-\frac{100}{2} - 100\ln 2 + 25(2)\right) - \left(-\frac{100}{1} - 100\ln 1 + 25(1)\right)\right]$$

$$= \pi\left[25 - 50 - 100\ln 2 + 50 + 100 - 25\right]$$

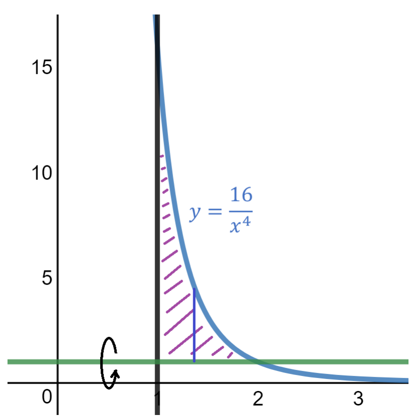
$$= \pi\left[100 - 100\ln 2\right]$$

$$= \boxed{100\pi(1 - \ln 2)}$$

2. Let S be the region bounded above by $x^4y = 16$, below by $y = 1$, on the left by $x = 1$, and on the right by $x = 2$.

(a) Find the volume of the solid generated by revolving S around the line $y = 1$.

Solution: Because we are revolving about the line $y = 1$, we are revolving about a horizontal line, which means our radius function **must be** a function of x. Given $x^4y = 16$, we solve for y to get $y = \dfrac{16}{x^4}$.



Now, the radius is given by $\dfrac{16}{x^4} - 1$ (which is the difference between the function and the line about which we are revolving) and the bounds are $1 \le x \le 2$. Note that we don't need to use the washer method here because there is no space between the region and where we are revolving.

So our volume is given by

$$\text{Volume} = \int_1^2 \pi\left(\frac{16}{x^4} - 1\right)^2\,dx$$

$$= \pi \int_1^2 \left(\frac{256}{x^8} - \frac{32}{x^4} + 1 \right) dx$$

$$= \pi \left[-\frac{256}{7x^7} + \frac{32}{3x^3} + x \right]_1^2$$

$$= \pi \left[\left(-\frac{256}{7(2)^7} + \frac{32}{3(2)^3} + 2 \right) - \left(-\frac{256}{7(1)^7} + \frac{32}{3(1)^3} + 1 \right) \right]$$

$$= \boxed{\frac{587\pi}{21}}$$

(b) Find the volume of the solid generated by revolving S around the line $x = 2$.

Solution: Here, we are revolving around a vertical line which means that our radius must be a function of y. We solve for x:

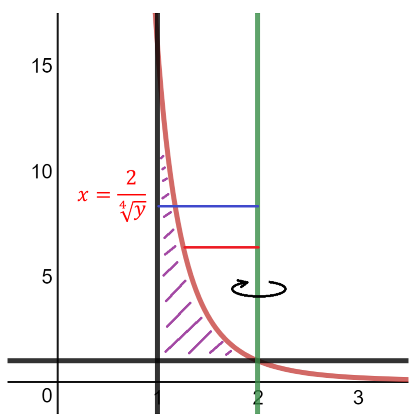
$$x^4 y = 16$$

$$\Rightarrow \quad x^4 = \frac{16}{y}$$

$$\Rightarrow \quad x = \sqrt[4]{\frac{16}{y}}$$

$$\Rightarrow \quad x = \frac{2}{y^{1/4}}$$

Thus, our graph looks like



This **does** require the washer method because there is a "gap" between the region and the line about which we are revolving. This means

$$\text{Outer Radius} : x = 2 - 1$$

$$\text{Inner Radius} : x = 2 - \frac{2}{y^{1/4}}$$

$$\text{Bounds} : 1 \le y \le 16$$

Therefore, our volume is given by

$$\text{Volume} = \int_1^{16} \pi \left[(2-1)^2 - \left(2 - \frac{2}{y^{1/4}}\right)^2 \right] dy$$

$$= \int_1^{16} \pi \left[(1)^2 - \left(2 - \frac{2}{y^{1/4}}\right)^2 \right] dy$$

$$= \pi \int_1^{16} \left[1 - \left(4 + \frac{4}{y^{1/2}} - \frac{8}{y^{1/4}}\right) \right] dy$$

$$= \pi \int_1^{16} \left(-3 - \frac{4}{y^{1/2}} + \frac{8}{y^{1/4}} \right) dy$$

$$= \pi \left[-3y - 8y^{1/2} + \frac{32}{3} y^{3/4} \right]_1^{16}$$

$$= \pi \left[-3(16) - 8(16)^{1/2} + \frac{32}{3}(16)^{3/4} - \left(-3(1) - 8(1)^{1/2} + \frac{32}{3}(1)^{3/4} \right) \right]$$

$$= \pi \left[-48 - 8(4) + \frac{32}{3}(2)^3 + 3 + 8 - \frac{32}{3} \right]$$

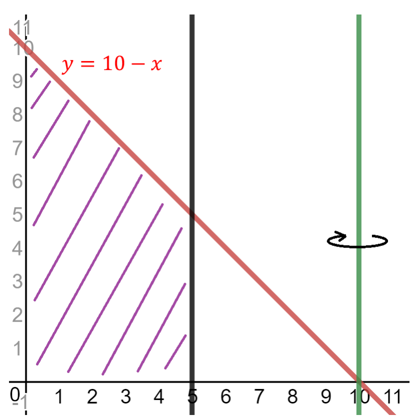
$$= \boxed{\frac{17\pi}{3}}$$

3. Find the volume of the solid generated by the region enclosed by

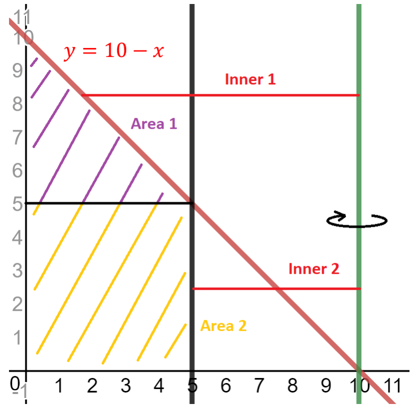
$$y = 10 - x, \quad y = 0, \quad x = 0, \quad x = 5$$

revolved about the line $x = 10$.

Solution: As usual, we sketch a graph of this region.



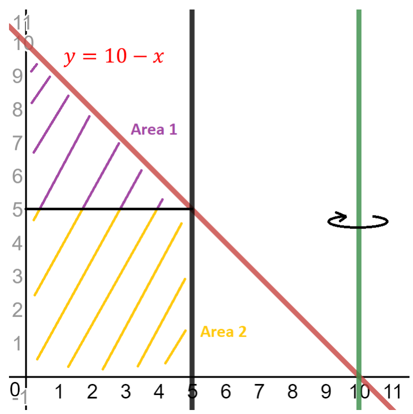
This is certainly a problem involving the washer method since there is a gap between the region and where we are revolving. Moreover, the inner radius differs depending on what y-value we choose. Thus, we need to divide this region into two areas:



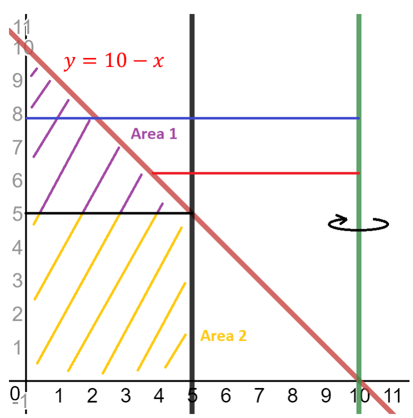
We also want to note that because we are revolving about a vertical line, we need to have radius functions as functions of y. So, we solve for x:

$$y = 10 - x \;\Rightarrow\; x = 10 - y.$$

Our picture is then



For Area 1, we see that

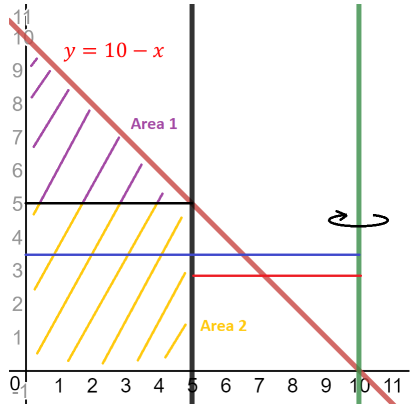


$$\text{Outer Radius} : x = 10 - 0$$

$$\text{Inner Radius} : x = 10 - (10 - y)$$

$$\text{Bounds} : 5 \leq y \leq 10$$

and for Area 2,



$$\text{Outer Radius}: x = 10 - 0$$

$$\text{Inner Radius}: x = 10 - 5$$

$$\text{Bounds}: 0 \leq y \leq 5$$

We put this together to compute the volume:

$$\text{Volume} = \underbrace{\int_0^5 \pi\left[(10-0)^2 - (10-5)^2\right]\,dy}_{\text{Area 2}} + \underbrace{\int_5^{10} \pi\left[(10-0)^2 - (10-(10-y))^2\right]\,dy}_{\text{Area 1}}$$

$$= \int_0^5 \pi\left[(10)^2 - (5)^2\right]\,dy + \int_5^{10} \pi\left[(10)^2 - (y)^2\right]\,dy$$

$$= \pi\left[\int_0^5 (100-25)\,dy + \int_5^{10} \left(100 - y^2\right)\,dy\right]$$

$$= \pi\left[\int_0^5 (75)\,dy + \int_5^{10} (100 - y^2)\,dy\right]$$

$$= \pi\left[75y\Big|_0^5 + 100y - \frac{1}{3}y^3\Big|_5^{10}\right]$$

$$= \pi\left[75(5) + \left(100(10) - \frac{1}{3}(10)^3 - \left(100(5) - \frac{1}{3}(5)^3\right)\right)\right]$$

$$= \pi\left[375 + \frac{625}{3}\right]$$

$$= \boxed{\frac{1750\pi}{3}}$$

NOTE 40. Revolving about a

- **horizontal line** — function of x and bounds in x
- **vertical line** — function of y and bounds in y

2. Additional Examples

EXAMPLES.

1. A propane tank is in the shape that is generated by revolving the region enclosed by the right half of the graph of

$$x^2 + 4y^2 = 16 \text{ and the } y\text{-axis}$$

about the y-axis. If x and y are measured in meters, find the depth of the propane in the tank when it is filled to one-quarter of the tank's volume. Round you answer to 3 decimal places.

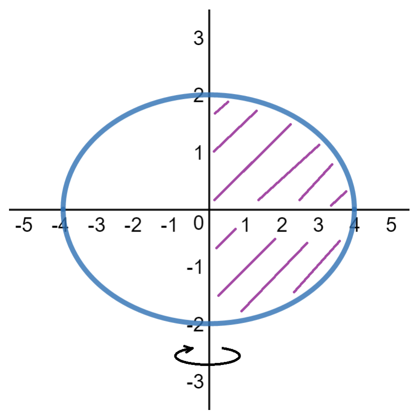
Solution: Our first step is to put the given equation into a more useful form. Divide through by 16 on both sides to get

$$\frac{x^2}{16} + \frac{4y^2}{16} = 1 \qquad \Longleftrightarrow \qquad \frac{x^2}{16} + \frac{y^2}{4} = 1.$$

This ellipse passes through (see note (37))

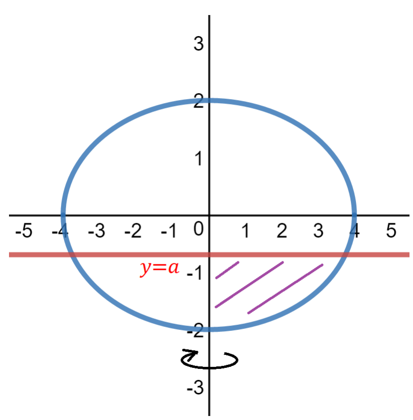
$$(4, 0), \quad (-4, 0), \quad (0, 2), \quad (0, -2).$$

Thus, our picture looks like



The purple area is the region enclosed by the ellipse and the y-axis.

Now, what are we asked to find? We are asked to find the horizontal line $y = a$ such that revolving the region enclosed by $\frac{x^2}{16} + \frac{y^2}{4} = 1$, the y-axis, **and** the line $y = a$ has only a fourth of the volume as the original tank.



Let's first determine the total volume of the tank. Since we are rotating around the y-axis, our function needs to be in the form $x =$ something. So

$$\begin{aligned} & \frac{x^2}{16} + \frac{y^2}{4} = 1 \\ \Rightarrow \quad & x^2 + 4y^2 = 16 \\ \Rightarrow \quad & x^2 = 16 - 4y^2 \\ \Rightarrow \quad & x = \sqrt{16 - 4y^2}. \end{aligned}$$

We take the positive root because we are considering only $x \geq 0$. Hence, the **total volume** of the tank is given by

$$\int_{-2}^{2} \pi \left(\sqrt{16 - 4y^2} \right)^2 \, dy.$$

Integrating:

$$\begin{aligned} \int_{-2}^{2} \pi \left(\sqrt{16 - 4y^2} \right)^2 \, dy &= \int_{-2}^{2} \pi (16 - 4y^2) \, dy \\ &= \pi \left(16y - \frac{4}{3} y^3 \right) \Big|_{-2}^{2} \\ &= \pi \left[16(2) - \frac{4}{3}(2)^3 - \left(16(-2) - \frac{4}{3}(-2)^3 \right) \right] \\ &= \pi \left[32 - \frac{32}{3} + 32 - \frac{32}{3} \right] \\ &= \pi \left[64 - \frac{64}{3} \right] \\ &= \frac{128\pi}{3}. \end{aligned}$$

But we aren't done yet. We don't want the *total volume*, we want **one-quarter of the volume**. This is where the line $y = a$ comes in. We need to find the a that solves the following equation:

$$\int_{-2}^{a} \pi \left(\sqrt{16 - 4y^2} \right)^2 \, dy = \underbrace{\frac{1}{4} \int_{-2}^{2} \pi \left(\sqrt{16 - 4y^2} \right)^2 \, dy}_{\text{one-quarter of the volume}}.$$

By above, we know that

$$\frac{1}{4} \int_{-2}^{2} \pi \left(\sqrt{16 - 4y^2} \right)^2 \, dy = \frac{1}{4} \left(\frac{128\pi}{3} \right) = \frac{32\pi}{3}.$$

Hence, we solve

$$\frac{32\pi}{3} = \int_{-2}^{a} \pi \left(\sqrt{16 - 4y^2} \right)^2 \, dy$$

for a.

So,

$$\begin{aligned}
\frac{32\pi}{3} &= \int_{-2}^{a} \pi \left(\sqrt{16-4y^2}\right)^2 \, dy \\
&= \pi \left(16y - \frac{4}{3}y^3\right)\Big|_{-2}^{a} \\
&= \pi \left[16a - \frac{4}{3}a^3 - \left(16(-2) - \frac{4}{3}(-2)^3\right)\right] \\
&= \pi \left[-\frac{4}{3}a^3 + 16a + 32 - \frac{32}{3}\right] \\
&= \pi \left[-\frac{4}{3}a^3 + 16a + \frac{64}{3}\right] \\
\Rightarrow \quad \frac{32}{3} &= -\frac{4}{3}a^3 + 16a + \frac{64}{3} \\
\Rightarrow \quad 0 &= -\frac{4}{3}a^3 + 16a + \frac{32}{3}
\end{aligned}$$

This is a cubic equation which you can't factor and so you can't solve it by hand. Instead, we find its root via a graphing calculator or Wolfram Alpha. Here, we have three possible solutions for a:

$$\begin{aligned}
a &\approx -3.0641778 \\
a &\approx -0.6945927 \\
a &\approx 3.7587705
\end{aligned}$$

We use the approximate forms here because each exact form here has an imaginary number in it and we don't address those in this class.

We need to think about which of these answers makes sense for how we have setup our problem: $a \approx -3.0641778$ and $a \approx 3.7587705$ are outside our bounds for y (because we are only looking at $-2 \leq y \leq 2$), so we throw those out. Thus, we are left with $a \approx -0.6945927$.

But how does it make sense for the depth from the bottom of the tank to be a negative number? It doesn't. However, notice that a is **not** the depth from the bottom of the tank. a is just height of a line. The depth from the bottom of the tank is the **difference** between $a = -0.6945927$ and the bottom of the tank, $y = -2$. Hence, our answer is

$$-0.6945927 - (-2) = 2 - 0.6945927 \approx \boxed{1.305 \text{ meters}}.$$

Lesson 15: Improper Integrals

1. Motivation

After practicing integration through a variety of applications, we return to some more theoretical aspects of the topic. Improper integrals address some of the deficiencies of typical definite integration. Until now, we limited ourselves to intervals of finite length and a function defined everywhere on that interval (i.e., the function made sense for every point in the interval). Improper integrals allow us to address (1) where a function might exist for very large numbers or (2) if the function doesn't exist at a point in the interval.

Ex 1. Suppose we are modeling how certain particles decay over time and we know that the energy given off by the particles at any time a is modeled by

$$\int_0^a 10e^{-10t}\, dt.$$

We might ask: how much energy is given off from now until the end of time? The time a would have to keep getting larger and larger, which we might write as $a \to \infty$. How does theis affect the integral?

$$\lim_{a\to\infty}\int_0^a 10e^{-10t}\, dt =: \int_0^\infty 10e^{-10t}\, dt.$$

This is an improper integral, the improper-ness coming from the fact that $[0, \infty)$ is an interval of infinite length. The symbol ":=" means "is defined to be".

Improper integrals **always** involve limits.

Comments on ∞:

- ∞ is **not** a number, **it is an upper bound**. All this means is that ∞ is larger than **every** real number.
- $\infty+(-\infty)$ is **undefined** — this means there is no consistent way to define what $\infty + (-\infty)$ should be.[1] If you are taking the limit and you get to the point where you have

$$\infty + (-\infty) \quad \text{or} \quad -\infty + \infty,$$

  then you have to go back and redo the limit.
- $\infty + 7 = \infty, 3\infty = \infty, -17\infty = -\infty$

Takeaway: ∞ is essentially different than finite things and so it should not be treated like finite things

2. Review of Basic Limits

Let $f(x)$ be a function and a some real number. Recall that

$\lim_{x\to a^+} f(x)$ is a **right hand limit** where we only consider $x > a$

$\lim_{x\to a^-} f(x)$ is a **left hand limit** where we only consider $x < a$

We say a limit **exists** if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

We call this shared value $\lim_{x\to a} f(x)$. If $f(x)$ is continuous at a (which, for this class, will more or less mean a is in the domain of f), then

$$\lim_{x\to a} f(x) = f(a).$$

3. Improper Integrals

EXAMPLES.

1. Compute $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

Solution: Improper integrals only make sense using limits. We make the following definition:

$$\int_1^\infty \frac{1}{x}\,dx := \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx.$$

Hence, improper integrals are done in two parts: we first evaluate

$$\int_1^t \frac{1}{x}\,dx$$

and second take the limit as $t \to \infty$.

Write

$$\begin{aligned}\int_1^t \frac{1}{x}\,dx &= \ln|x|\Big|_1^t \\ &= \ln t - \underbrace{\ln 1}_{0} \\ &= \ln t\end{aligned}$$

where we drop the $|\cdot|$ because we are assuming that $t > 0$.

---

[1]Suppose we let $\infty + (-\infty) := 0$. What issues might this raise? Well, $7 + \infty = \infty$ but

$$\begin{aligned} 7 + \infty + (-\infty) &= \infty + (-\infty) \\ \Rightarrow \quad 7 + (0) &= 0 \\ \Rightarrow \quad 7 &= 0 \end{aligned}$$

So this doesn't make any sense.

Next, we take the limit

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx\\ &= \lim_{t\to\infty} \ln t\\ &= \boxed{\infty}\end{aligned}$$

We say this integral diverges.

Definition 41. An integral **diverges** if its limit is $\pm\infty$ or DNE.

Note 42. Take time to review the techniques of finding limits of functions.

Basic Limits: for $n > 0$, $k > 0$,

$$\begin{array}{r|l} \displaystyle\lim_{t\to\infty}\frac{1}{t^n} = 0 & \displaystyle\lim_{t\to\infty}\frac{1}{e^t} = 0 \\ \displaystyle\lim_{t\to\infty} t^n = \infty & \displaystyle\lim_{t\to\infty}\ln t = \infty \\ \displaystyle\lim_{t\to\infty}\frac{t^n}{e^{tk}} = 0 & \displaystyle\lim_{t\to 0^+}\ln t = -\infty \\ \displaystyle\lim_{t\to\infty}\frac{e^{tk}}{t^n} = \infty & \displaystyle\lim_{t\to\infty}\frac{1}{\ln x} = 0 \end{array}$$

$$\lim_{t\to a}(cf(t)+g(t)) = c\left(\lim_{t\to a} f(t)\right) + \lim_{t\to a} g(t)$$

2. Find $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

Solution: By definition,

$$\int_1^\infty \frac{1}{x^2} := \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx.$$

So, we compute $\displaystyle\int_1^t \frac{1}{x^2}\,dx$ and then take its limit as $t\to\infty$. Write

$$\begin{aligned}\int_1^t \frac{1}{x^2}\,dx &= \int_1^t x^{-2}\,dx\\ &= \frac{1}{-2+1}x^{-2+1}\Big|_1^t\\ &= -x^{-1}\Big|_1^t\\ &= -\frac{1}{x}\Big|_1^t\end{aligned}$$

$$= -\frac{1}{t} - \left(-\frac{1}{1}\right)$$
$$= -\frac{1}{t} + 1$$

Next, we take the limit as $t \to \infty$:

$$\lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = -(0) + 1 = 1.$$

Therefore, we say the integral converges and write

$$\int_1^\infty \frac{1}{x^2}\,dx = \boxed{1}.$$

DEFINITION 43. An integral **converges** if its limit is a real number.

3. Determine if

$$\int_0^\infty -3x^2 e^{-x^3}\,dx$$

converges or diverges. If it converges, find its value.

Solution: We know

$$\int_0^\infty -3x^2 e^{-x^3}\,dx := \lim_{t\to\infty} \underbrace{\int_0^t -3x^2 e^{-x^3}\,dx}_{u\text{-sub}}.$$

Let $u = -x^3$, then $du = -3x^2$. Further,

$$u(0) = -(0)^3 = 0$$
$$u(t) = -(t)^3 = -t^3$$

So

$$\int_0^t -3x^2 e^{-x^3}\,dx = \int_0^{-t^3} e^u\,du$$
$$= e^u\Big|_0^{-t^3}$$
$$= e^{-t^3} - e^0$$
$$= e^{-t^3} - 1$$
$$= \frac{1}{e^{t^3}} - 1$$

Therefore,

$$\int_0^{\infty} -3x^2e^{-x^3}\,dx = \lim_{t\to\infty}\int_0^t -3x^2e^{-x^3}\,dx = \lim_{t\to\infty}\left(\frac{1}{e^{t^3}}-1\right) = 0-1=-1$$

So the interval converges and its value is -1.

4. Determine if

$$\int_1^2 \frac{1}{x-1}\,dx$$

converges or diverges. If it converges, find its value.

Solution: This is different from Examples 1-3 because we are *not* dealing with an interval of infinite length. How is this an improper integral? Well, the function $\frac{1}{x-1}$ **doesn't exist** at $x=1$ (because then we would be dividing by 0). For a definite integral to make sense the function **must exist** on the *entire* interval. So what do we do here? We take a limit as $x\to 1$.

Write

$$\int_1^2 \frac{1}{x-1}\,dx := \lim_{s\to 1^+}\int_s^2 \frac{1}{x-1}\,dx.$$

Recall that $\lim_{s\to 1^+}$ means we are looking at numbers that are getting very close to 1 *but* which are all **bigger** than 1. Why do we care if our numbers are bigger than 1? Our interval is $[1,2]$, which means we don't care about anything that is **less** than 1.

As before, we first evaluate $\int_s^2 \frac{1}{x-1}\,dx$ and then take the limit.

We have

$$\begin{aligned}\int_s^2 \frac{1}{x-1}\,dx &= \ln|x-1|\Big|_s^2 \\ &= \ln|2-1| - \ln|s-1| \\ &= \underbrace{\ln(1)}_{0} - \ln|s-1| \\ &= -\ln(s-1)\end{aligned}$$

Why do we drop the $|\cdot|$? Because we are **only interested in** $s>1$, which means that $s-1>0$.

Now, we take the limit:

$$\begin{aligned}\int_1^2 \frac{1}{x-1}\,dx &= \lim_{s\to 1^+}\int_s^2 \frac{1}{x-1}\,dx \\ &= -\lim_{s\to 1^+}\ln(s-1)\end{aligned}$$

$$\begin{aligned}
&\overset{(2)}{=} -\lim_{t\to 0^+} \ln(t) \text{ where } t = s-1 \\
&= -(-\infty) \\
&= \infty
\end{aligned}$$

But as $t \to 0$, $\ln(t) \to -\infty$.

Therefore, we conclude the integral diverges.

5. Determine if

$$\int_{10}^{\infty} \frac{1}{x(\ln x)^2}\, dx$$

converges or diverges. If it converges, find its value.

Solution: We write

$$\int_{10}^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{t\to\infty} \underbrace{\int_{10}^{t} \frac{1}{x(\ln x)^2}\, dx}_{u\text{-sub}}.$$

Let $u = \ln x$, then $du = \dfrac{1}{x}\, dx$ with

$$\begin{aligned}
u(10) &= \ln(10) \\
u(t) &= \ln t
\end{aligned}$$

So

$$\begin{aligned}
\int_{10}^{t} \frac{1}{x(\ln x)^2}\, dx &= \int_{\ln 10}^{\ln t} \frac{1}{u^2}\, du \\
&= \int_{\ln 10}^{\ln t} u^{-1}\, du \\
&= \frac{1}{-2+1} u^{-2+1} \Big|_{\ln 10}^{\ln t} \\
&= -u^{-1} \Big|_{\ln 10}^{\ln t} \\
&= -\frac{1}{u} \Big|_{\ln 10}^{\ln t} \\
&= -\frac{1}{\ln t} - \left(-\frac{1}{\ln 10}\right) \\
&= \frac{1}{\ln 10} - \frac{1}{\ln t}.
\end{aligned}$$

(2) This follows because as $s \to 1^+$, $t = s - 1 \to 0^+$.

Taking the limit, we see that

$$\lim_{t\to\infty}\left(\frac{1}{\ln 10}-\frac{1}{\ln t}\right)=\frac{1}{\ln 10}-0=\frac{1}{\ln 10}.$$

We conclude the integral converges and

$$\int_{10}^{\infty}\frac{1}{x(\ln x)^2}\,dx=\boxed{\frac{1}{\ln 10}}.$$

6. Compute $\displaystyle\int_0^\infty \frac{x}{e^x}\,dx$.

Write

$$\int_0^\infty \frac{x}{e^x}\,dx=\int_0^\infty xe^{-x}\,dx=\lim_{t\to\infty}\int_0^t xe^{-x}\,dx.$$

This is an integration by parts integral. By LIATE,

$$u=x \qquad dv=e^{-x}\,dx$$

$$du=dx \qquad v=-e^{-x}$$

Hence,

$$\begin{aligned}\int_0^t xe^{-x}\,dx &= -xe^{-x}\Big|_0^t-\int_0^t(-e^{-x})\,dx\\ &= -xe^{-x}\Big|_0^t-e^{-x}\Big|_0^t\\ &= (-xe^{-x}-e^{-x})\Big|_0^t\\ &= -te^{-t}-e^{-t}-\big(-0(e^{-0})-e^{-0}\big)\\ &= -te^{-t}-e^{-t}+1\end{aligned}$$

Taking the limit,

$$\lim_{t\to\infty}(-te^{-t}-e^{-t}+1)=\lim_{t\to\infty}\left(-\frac{t}{e^t}-\frac{1}{e^t}+1\right)=0+0+1=1.$$

We conclude

$$\int_0^\infty \frac{x}{e^x}\,dx=\boxed{1}.$$

4. Additional Examples

EXAMPLES.

1. Compute $\displaystyle\int_1^\infty \frac{1}{\sqrt[6]{x}}\,dx$.

Solution: By definition,

$$\int_1^\infty \frac{1}{\sqrt[6]{x}}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt[6]{x}}\,dx.$$

Now, we integrate $\displaystyle\int_1^t \frac{1}{\sqrt[6]{x}}\,dx$:

$$\begin{aligned}\int_1^t \frac{1}{\sqrt[6]{x}}\,dx &= \int_1^t \frac{1}{x^{1/6}}\,dx \\ &= \int_1^t x^{-1/6}\,dx \\ &= \left(\frac{1}{-1/6+1}\right) x^{-1/6+1}\Big|_1^t \\ &= \left(\frac{1}{5/6}\right) x^{5/6}\Big|_1^t \\ &= \left(\frac{6}{5}\right) x^{5/6}\Big|_1^t \\ &= \frac{6}{5}t^{5/6} - \frac{6}{5}(1^{5/6}) \\ &= \frac{6}{5}t^{5/6} - \frac{6}{5}\end{aligned}$$

Next, we take the limit as $t \to \infty$:

$$\lim_{t\to\infty}\left(\frac{6}{5}t^{5/6} - \frac{6}{5}\right) = \frac{5}{6}\cdot\infty - \frac{6}{5} = \infty.$$

Therefore, we conclude that $\displaystyle\int_1^\infty \frac{1}{\sqrt[6]{x}}\,dx$ diverges.

2. Determine if

$$\int_0^\pi \sec^2(x)\,dx$$

converges or diverges. If it converges, find its value.

Solution: Again, we are looking at an interval of finite length. Why is this an improper integral? Recall that $\sec(x) = \dfrac{1}{\cos(x)}$ and that $\cos\left(\dfrac{\pi}{2}\right) = 0$.

Thus, $\sec^2(x)$ does **not** exist on *all* of the interval $[0, \pi]$. We address this by breaking the interval into two halves: $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$ because $\sec^2(x)$ exists on both of these intervals sans the point $\frac{\pi}{2}$.

We have

$$\int_0^{\pi} \sec^2(x)\,dx = \int_0^{\pi/2} \sec^2(x)\,dx + \int_{\pi/2}^{\pi} \sec^2(x)\,dx$$

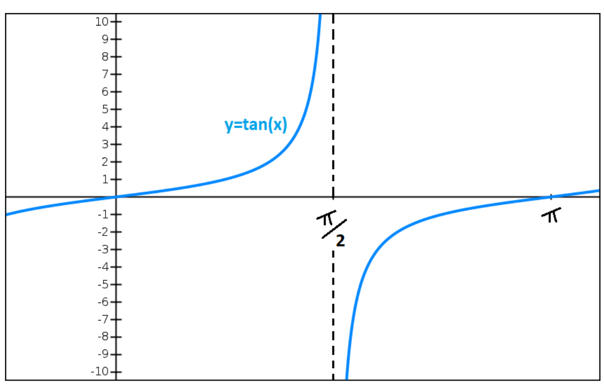
where **both** integrals on the right are improper integrals. Hence,

$$\int_0^{\pi/2} \sec^2(x)\,dx + \int_{\pi/2}^{\pi} \sec^2(x)\,dx := \lim_{s\to\frac{\pi}{2}^-} \int_0^{s} \sec^2(x)\,dx + \lim_{t\to\frac{\pi}{2}^+} \int_t^{\pi} \sec^2(x)\,dx.$$

The antiderivative of $\sec^2(x)$ is $\tan(x)$. So

$$\begin{aligned}
&\lim_{s\to\frac{\pi}{2}^-} \int_0^{s} \sec^2(x)\,dx + \lim_{t\to\frac{\pi}{2}^+} \int_t^{\pi} \sec^2(x)\,dx \\
&= \lim_{s\to\frac{\pi}{2}^-} \tan(x)\Big|_0^s + \lim_{t\to\frac{\pi}{2}^+} \tan(x)\Big|_t^{\pi} \\
&= \lim_{s\to\frac{\pi}{2}^-} (\tan(s) - \underbrace{\tan(0)}_{0}) + \lim_{t\to\frac{\pi}{2}^+} (\underbrace{\tan(\pi)}_{0} - \tan(t)) \\
&= \lim_{s\to\frac{\pi}{2}^-} \tan(s) - \lim_{t\to\frac{\pi}{2}^+} \tan(t).
\end{aligned}$$

To determine what $\tan(x)$ is doing at $x = \frac{\pi}{2}$, we consider its graph



Interpreting the graph, we conclude that

$$\lim_{s\to\frac{\pi}{2}^-} \tan(s) = \infty \quad \text{and} \quad \lim_{t\to\frac{\pi}{2}^+} \tan(t) = -\infty.$$

Putting this together,

$$\int_0^{\pi} \sec^2(x)\,dx = \lim_{s\to\frac{\pi}{2}^-} \tan(s) - \lim_{t\to\frac{\pi}{2}^+} \tan(t) = \infty - (-\infty) = \infty + \infty = \infty.$$

We conclude that the integral diverges.

3. Evaluate $\displaystyle\int_1^\infty \frac{10x^2}{(4x^3+4)^{5/2}}\,dx$.

Solution: Write

$$\int_1^\infty \frac{10x^2}{(4x^3+4)^{5/2}}\,dx = \lim_{t\to\infty}\int_1^t \frac{10x^2}{(4x^3+4)^{5/2}}\,dx.$$

We integrate via u-sub. Let $u = 4x^3+4$, then $\dfrac{du}{dx} = 12x^2 \;\Rightarrow\; dx = \dfrac{du}{12x^2}$. Then

$$\begin{aligned}
\int_1^t \frac{10x^2}{(4x^3+4)^{5/2}}\,dx &= \int_{u(1)}^{u(t)} \frac{10x^2}{u^{5/2}} \underbrace{\left(\frac{du}{12x^2}\right)}_{dx} \\
&= \int_{u(1)}^{u(t)} \frac{5}{6u^{5/2}}\,du \\
&= \int_{u(1)}^{u(t)} \frac{5}{6}u^{-5/2}\,du \\
&= \frac{5}{6}\left(\frac{1}{-5/2+1}\right) u^{-5/2+1}\Big|_{u(1)}^{u(t)} \\
&= \frac{5}{6}\left(\frac{1}{-3/2}\right) u^{-3/2}\Big|_{u(1)}^{u(t)} \\
&= \frac{5}{6}\left(-\frac{2}{3}\right) u^{-3/2}\Big|_{u(1)}^{u(t)} \\
&= -\frac{5}{9}u^{-3/2}\Big|_{u(1)}^{u(t)} \\
&= -\frac{5}{9}(4x^3+4)^{-3/2}\Big|_1^t \\
&= -\frac{5}{9}(4t^3+4)^{-3/2} - \left[-\frac{5}{9}(4+4)^{-3/2}\right] \\
&= -\frac{5}{9(4t^3+4)^{3/2}} + \frac{5}{9(8)^{3/2}}
\end{aligned}$$

Next, we take the limit:

$$\lim_{t\to\infty}\left[-\frac{5}{9(4t^3+4)^{3/2}} + \frac{5}{9(8)^{3/2}}\right] = 0 + \frac{5}{9(8)^{3/2}} = \frac{5}{9(8)^{3/2}}.$$

Therefore, $\displaystyle\int_1^\infty \frac{10x^2}{(4x^3+4)^{5/2}}\,dx$ $\boxed{\text{converges}}$ and

$$\int_1^\infty \frac{10x^2}{(4x^3+4)^{5/2}}\,dx = \boxed{\frac{5}{9(8)^{3/2}}}.$$

4. Evaluate $\displaystyle\int_1^\infty \frac{6e^{-\sqrt{x}}}{7\sqrt{x}}\,dx$.

Solution: We first simplify the function by replacing the roots with fractional exponents and doing a slight rewrite:

$$\frac{6e^{-\sqrt{x}}}{7\sqrt{x}} = \frac{6e^{-x^{1/2}}}{7x^{1/2}} = \frac{6}{7}e^{-x^{1/2}}x^{-1/2}.$$

Next, we separate our improper integral into a limit and a proper integral:

$$\int_1^\infty \frac{6}{7}e^{-x^{1/2}}x^{-1/2}\,dx = \lim_{t\to\infty}\int_1^t \frac{6}{7}e^{-x^{1/2}}x^{-1/2}\,dx.$$

We integrate via u-sub. Let $u = -x^{1/2}$, then $\dfrac{du}{dx} = -\dfrac{1}{2}x^{-1/2} \Rightarrow dx = -2x^{1/2}du$. Substituting and then integrating,

$$\begin{aligned}
\int_1^t \frac{6}{7}e^{-x^{1/2}}x^{-1/2}\,dx &= \int_{u(1)}^{u(t)} \frac{6}{7}e^u x^{-1/2}\underbrace{(-2x^{1/2}\,du)}_{dx} \\
&= \int_{u(1)}^{u(t)} -\frac{12}{7}e^u\,du \\
&= -\frac{12}{7}e^u\Big|_{u(1)}^{u(t)} \\
&= -\frac{12}{7}e^{-x^{1/2}}\Big|_1^t \\
&= -\frac{12}{7}\left[e^{-t^{1/2}} - e^{-1^{1/2}}\right] \\
&= -\frac{12}{7}\left[\frac{1}{e^{t^{1/2}}} - \frac{1}{e}\right] \\
&= -\frac{12}{7e^{t^{1/2}}} + \frac{12}{7e}
\end{aligned}$$

Taking the limit, we have

$$\lim_{t\to\infty}\left[-\frac{12}{7e^{t^{1/2}}} + \frac{12}{7e}\right] = 0 + \frac{12}{7e} = \frac{12}{7e}.$$

Hence, the integral converges and

$$\int_1^\infty \frac{6}{7} e^{-x^{1/2}} x^{-1/2}\, dx = \boxed{\frac{12}{7e}}.$$

5. Evaluate $\displaystyle\int_4^6 \frac{1}{\sqrt[7]{x-4}}\, dx$.

Solution: Although our bounds don't include an infinity, this is *still* an improper integral because the function we are integrating does not exist at $x = 4$. However, before we attempt to integrate, we'll rewrite the function so it is easier to manipulate:

$$\frac{1}{\sqrt[7]{x-4}} = \frac{1}{(x-4)^{1/7}} = (x-4)^{-1/7}.$$

So our improper integral becomes

$$\int_4^6 \frac{1}{\sqrt[7]{x-4}}\, dx = \lim_{s \to 4^+} \int_s^6 (x-4)^{-1/7}\, dx.$$

Next, we integrate:

$$\begin{aligned}
\int_s^6 (x-4)^{-1/7}\, dx &= \left(\frac{1}{-1/7+1}\right)(x-4)^{-1/7+1}\Big|_s^6 \\
&= \left(\frac{1}{6/7}\right)(x-4)^{6/7}\Big|_s^6 \\
&= \frac{7}{6}(x-4)^{6/7}\Big|_s^6 \\
&= \frac{7}{6}(6-4)^{6/7} - \frac{7}{6}(s-4)^{6/7} \\
&= \frac{7(2^{6/7})}{6} - \frac{7}{6}(s-4)^{6/7}
\end{aligned}$$

Finally, we apply our limit:

$$\lim_{s \to 4^+}\left[\frac{7(2^{6/7})}{6} - \frac{7}{6}(s-4)^{6/7}\right] = \frac{7(2^{6/7})}{6} - 0 = \frac{7(2^{6/7})}{6}.$$

We conclude the integral converges and

$$\int_4^6 \frac{1}{\sqrt[7]{x-4}}\, dx = \boxed{\frac{7(2^{6/7})}{6}}.$$

6. Evaluate $\displaystyle\int_1^\infty 10(x-1)e^{-7x}\, dx$.

Solution: Write

$$\int_1^\infty 10(x-1)e^{-7x}\,dx = \lim_{t\to\infty}\int_1^t 10(x-1)e^{-7x}\,dx.$$

This integral requires integration by parts.

By LIATE, take $u = x - 1$, then our table becomes

$$u = x - 1 \qquad dv = 10e^{-7x}\,dx$$
$$du = dx \qquad v = -\frac{10}{7}e^{-7x}$$

So,

$$\int_1^t 10(x-1)e^{-7x}\,dx = \underbrace{(x-1)}_{u}\underbrace{\left(-\frac{10}{7}e^{-7x}\right)}_{v}\Bigg|_1^t - \int_1^t \underbrace{\left(-\frac{10}{7}e^{-7x}\right)}_{v}\underbrace{dx}_{du}$$

$$= -\frac{10}{7}(x-1)e^{-7x}\Bigg|_1^t + \frac{10}{7}\int_1^t e^{-7x}\,dx$$

$$= -\frac{10}{7}(x-1)e^{-7x}\Bigg|_1^t - \frac{10}{49}e^{-7x}\Bigg|_1^t$$

$$= -\frac{10}{7}(x-1)e^{-7x} - \frac{10}{49}e^{-7x}\Bigg|_1^t$$

$$= -\frac{10}{7}(t-1)e^{-7t} - \frac{10}{49}e^{-7t} - \left[-\frac{10}{7}(1-1)e^{-7(1)} - \frac{10}{49}e^{-7(1)}\right]$$

$$= -\frac{10(t-1)}{7e^{7t}} - \frac{10}{49e^{7t}} + \frac{10}{49e^{7}}$$

Taking the limit,

$$\lim_{t\to\infty}\left[-\frac{10(t-1)}{7e^{7t}} - \frac{10}{49e^{7t}} + \frac{10}{49e^{7}}\right] = 0 + 0 + \frac{10}{49e^7} = \frac{10}{49e^7}.$$

We conclude that the integral $\boxed{\text{converges}}$ and

$$\int_1^\infty 10(x-1)e^{-7x}\,dx = \boxed{\frac{10}{49e^7}}.$$

Lesson 16: Geometric Series and Convergence (I)

1. Introduction to Series

Series are just sums of things, like numbers or functions.

Ex 1.

$$\sum_{n=1}^{3}\frac{2^n+1}{3^n-2}=\underbrace{\frac{2^1+1}{3^1-2}}_{n=1}+\underbrace{\frac{2^2+1}{3^2-2}}_{n=2}+\underbrace{\frac{2^3+1}{3^3-2}}_{n=3}.$$

This is **summation notation**. The $\sum$ (called "sigma") means to add a bunch of things together. The n is called the **index** and is used to put an ordering on the sum (so that we can keep track of what we're adding together). The number under $\sum$ tells us *when* to start counting and the number above $\sum$ tells us *when* to stop counting.

We can also use series to talk about the sum of an infinite number of things.

Ex 2.

$$\sum_{n=0}^{\infty}\frac{1}{n+1}:=\lim_{t\to\infty}\sum_{n=0}^{t}\frac{1}{n+1},\quad \sum_{n=1}^{\infty}a_n:=\lim_{t\to\infty}\sum_{n=1}^{t}a_n$$

Definition 44. We say the n^{th} **partial sum** is the sum of the first n terms.

Ex 3. Consider $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}$

The 3^{rd} partial sum is

$$\underbrace{\frac{1}{0+1}+\frac{1}{1+1}+\frac{1}{2+1}}_{3\text{ terms}}=\frac{11}{6}$$

and the 5^{th} partial sum is

$$\underbrace{\frac{1}{0+1}+\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}+\frac{1}{4+1}}_{5\text{ terms}}=\frac{137}{60}.$$

Definition 45. We say a series **converges** if the partial sums limit to a finite number. We say the series **diverges** otherwise.

Series Facts:

(1) $c\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ca_n$

Ex: $3\sum_{n=1}^{\infty}\frac{1}{2^n} = \sum_{n=1}^{\infty}\frac{3}{2^n}$ and $\frac{1}{8}\sum_{n=17}^{\infty}\frac{3}{4^{n-3}} = \sum_{n=17}^{\infty}\frac{3}{8(4^{n-3})}$

(2) $\sum_{n=0}^{\infty} a_n = \sum_{n=m}^{\infty} a_{n-m}$, $\sum_{n=m}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+m}$

Ex:

- Given $\sum_{n=0}^{\infty}\frac{.6^n}{n+1}$, we make this series start at $n = 1$ by **subtracting** 1 from every instance of n:

$$\sum_{n=0}^{\infty}\frac{.6^n}{n+1} = \sum_{n=1}^{\infty}\frac{.6^{n-1}}{n}$$

- Given $\sum_{n=3}^{\infty}\frac{4(3^{2n})}{5^n}$, we make this series start at $n = 0$, by **adding** 3 from every instance of n:

$$\sum_{n=3}^{\infty}\frac{4(3^{2n})}{5^n} = \sum_{n=0}^{\infty}\frac{4(3^{2(n+3)})}{5^{n+3}}$$

2. Geometric Series

A **geometric series** is a series of the form

$$\sum_{n=m}^{\infty} cr^{n+k}.$$

Ex 4.

- Geometric Series: $\sum_{n=1}^{\infty}\frac{3}{7^{n-1}}$, $\frac{1}{8}\sum_{n=0}^{\infty}\frac{1}{2^n}$
- Not Geometric Series: $\sum_{n=1}^{\infty}\frac{1}{n^3}$, $\sum_{n=3}^{\infty}\frac{16(-1)^{n+1}n!}{4^n}$

Geometric series are special because we can actually compute what the infinite sum is (which is actually very difficult for any other type of series). In fact, we even have a formula for the geometric series: if $|r| < 1$ then

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}. \tag{11}$$

If $|r| \geq 1$, then the series diverges and we **cannot** apply this formula.

REMARK 46. Simply applying the formula doesn't mean the series converges. For example, we may apply equation (11) to the series $\sum_{n=0}^{\infty}(-1)^n$ but this series **diverges** because $|r| = |-1| = 1$.

NOTE 47. Make particular note of where the series starts and what power we are raising r to. To use equation (11), the geometric series **must look exactly like the LHS of equation (11)**.

EXAMPLES.

1. If $17 - \frac{34}{8} + \frac{51}{27} - \frac{68}{64} + \frac{85}{125} - \cdots$ continues as a pattern, write it in summation notation.

Solution: We think about the pattern in the numbers. Consider

$$\begin{aligned}
&17 - \frac{34}{8} + \frac{51}{27} - \frac{68}{64} + \frac{85}{125} - \cdots \\
&= 17\left(1 - \frac{2}{8} + \frac{3}{27} - \frac{4}{64} + \frac{5}{125} - \cdots\right) \\
&= 17\left(1 - \frac{2}{2^3} + \frac{3}{3^3} - \frac{4}{4^3} + \frac{5}{5^3} - \cdots\right) \\
&= 17\Bigg(\underbrace{\frac{(-1)^2}{1^3}}_{n=1} + \underbrace{\frac{2(-1)^3}{2^3}}_{n=2} + \underbrace{\frac{3(-1)^4}{3^3}}_{n=3} + \underbrace{\frac{4(-1)^5}{4^3}}_{n=4} + \underbrace{\frac{5(-1)^6}{5^3}}_{n=5} + \cdots\Bigg) \\
&= \boxed{17\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^3} \text{ or } 17\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}}.
\end{aligned}$$

2. Compute $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n$.

Solution: This is a geometric series but it is not in the correct form for us to apply equation (11) because n does not start at 0. We resolve this by playing around with the index:

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(-\frac{1}{2}\right)}_{c} \underbrace{\left(-\frac{1}{2}\right)^n}_{r^n} \leftarrow \text{Correct Form}$$

$$= \frac{-1/2}{1-(-1/2)}$$

$$= \frac{2}{2}\left(\frac{(-1/2)}{1+(1/2)}\right)$$

$$= \frac{-1}{2+1} = \boxed{-\frac{1}{3}}$$

3. Compute $\sum_{n=0}^{\infty} 4e^{-2n}$.

Solution: This is a geometric series, although it might not seem like it. We rewrite to make this fact more apparent. Consider

$$\sum_{n=0}^{\infty} 4e^{-2n} = \sum_{n=0}^{\infty} 4(e^{-2})^n.$$

Now, because $|e^{-2}| = \left|\frac{1}{e^2}\right| < 1$ (since $e^2 > 1$), this geometric series converges. By the geometric series formula,

$$\sum_{n=0}^{\infty} 4(e^{-2})^n = \frac{4}{1-e^{-2}} = \frac{4}{1-e^{-2}} \cdot \frac{e^2}{e^2} = \boxed{\frac{4e^2}{e^2-1}}.$$

4. Compute $\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n}$

Solution: This is also a geometric series, but not in the correct form to directly apply equation (11). We write

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{4^n} = \sum_{n=0}^{\infty} \frac{3^2 3^n}{4^n}$$

$$= \sum_{n=0}^{\infty} 9\left(\frac{3^n}{4^n}\right)$$

$$= \sum_{n=0}^{\infty} 9\left(\frac{3}{4}\right)^n$$

$$= \frac{9}{1-3/4} \text{ by the equation (11)}$$

$$= \frac{9}{1/4} = 4 \cdot 9 = \boxed{36}$$

5. Compute $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{(-3)^n} + \frac{6}{4^{n+1}} \right)$.

Solution: We tackle these sorts of problems by splitting up the summation. Write

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(-3)^n} &= \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \\
&= \frac{1}{1+1/3} = \frac{3}{4} \\
\sum_{n=0}^{\infty} \frac{6}{4^{n+1}} &= \sum_{n=0}^{\infty} \left(\frac{6}{4}\right)\left(\frac{1}{4^n}\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)\left(\frac{1}{4}\right)^n \\
&= \frac{3/2}{1-1/4} \\
&= \frac{3}{2}\left(\frac{4}{3}\right) = \frac{12}{6} = 2
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} \left(\frac{1}{(-3)^n} + \frac{6}{4^{n+1}} \right) = \sum_{n=0}^{\infty} \frac{1}{(-3)^n} + \sum_{n=0}^{\infty} \frac{6}{4^{n+1}} = \frac{3}{4} + 2 = \boxed{\frac{11}{4}}$$

6. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{3(-1)^n}{5^{2n}}$

Solution: Again, this is not in the correct form:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3(-1)^n}{(5^2)^n} &= \sum_{n=0}^{\infty} \frac{3(-1)^{n+1}}{(5^2)^{n+1}} \\
&= \sum_{n=0}^{\infty} 3\left(-\frac{1}{5^2}\right)^{n+1} \\
&= \sum_{n=0}^{\infty} 3\left(-\frac{1}{5^2}\right)\left(-\frac{1}{5^2}\right)^n \\
&= -\frac{3}{25}\left(\frac{1}{1+1/25}\right)
\end{aligned}$$

$$= -\frac{3}{25}\left(\frac{25}{26}\right)$$

$$= \boxed{-\frac{3}{26}}$$

7. Using summation notation, rewrite the number $6.3\overline{1}$.

Solution: This problem comes down to interpreting what decimals mean: $6.32\overline{1}$ has 6 ones, 3 tenths, 1 hundredths, 1 thousandths etc. We might even write this as

$$6.3\overline{1} = \underset{\substack{\uparrow \\ \text{ones}}}{6} + \underset{\substack{\uparrow \\ \text{tenths}}}{\frac{3}{10}} + \underset{\substack{\uparrow \\ \text{hundredths}}}{\frac{1}{100}} + \underset{\substack{\uparrow \\ \text{thousandths}}}{\frac{1}{1{,}000}} + \frac{1}{10{,}000} + \cdots$$

$$= 6 + \frac{3}{10} + \sum_{n=0}^{\infty} \frac{1}{10^{n+2}}$$

$$= \boxed{6 + \frac{3}{10} + \sum_{n=2}^{\infty} \frac{1}{10^n}}$$

3. Additional Examples

EXAMPLES.

1. Find the fifth partial sum of

$$\sum_{n=0}^{\infty} \frac{16(-1)^{n+1} n!}{4^n}.$$

Solution: The fifth partial sum is the sum of the **first five terms**. Consult Appendix (F) to check the meaning of $n!$.

Write

$$\frac{16(-1)^{0+1}0!}{4^0} + \frac{16(-1)^{1+1}1!}{4^1} + \frac{16(-1)^{2+1}2!}{4^2} + \frac{16(-1)^{3+1}3!}{4^3} + \frac{16(-1)^{4+1}4!}{4^4}$$

$$= \frac{16(-1)^1(1)}{1} + \frac{16(-1)^2(1)}{4} + \frac{16(-1)^3(2)}{16} + \frac{16(-1)^4(6)}{64} + \frac{16(-1)^5(24)}{256}$$

$$= -16 + 4 - 2 + \frac{3}{2} - \frac{3}{2}$$

$$= \boxed{-14}$$

2. Put $12 - \frac{24}{8} + \frac{36}{27} - \frac{48}{64} + \frac{60}{125} - \cdots$ into summation notation.

Solution: Observe that

$$12 = 12 \cdot 1, \ 24 = 12 \cdot 2, \ 36 = 12 \cdot 3, \ 48 = 12 \cdot 4, \ 60 = 12 \cdot 5$$

and

$$1 = 1^3,\ 8 = 2^3,\ 27 = 3^3,\ 64 = 4^3,\ 125 = 5^3.$$

Further, notice

$$1 = (-1)^{1+1},\ -1 = (-1)^{2+1},\ 1 = (-1)^{3+1},\ -1 = (-1)^{4+1},\ 1 = (-1)^{5+1}.$$

Putting this all together,

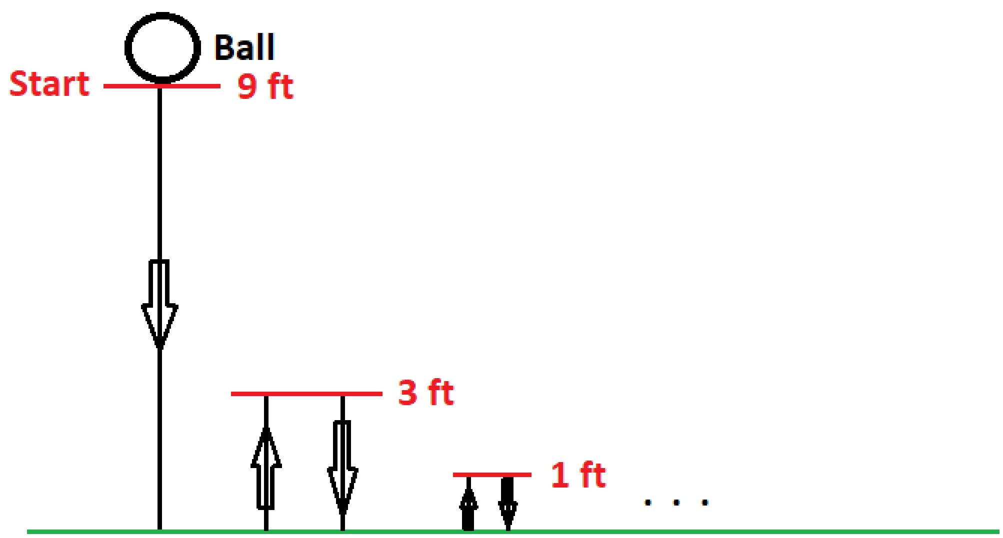
$$\begin{aligned}
&12 - \frac{24}{8} + \frac{36}{27} - \frac{48}{64} + \frac{60}{125} - \cdots \\
&= \frac{(-1)^{1+1}(12 \cdot 1)}{1^3} + \frac{(-1)^{2+1}(12 \cdot 2)}{2^3} + \frac{(-1)^{3+1}(12 \cdot 3)}{3^3} + \frac{(-1)^{4+1}(12 \cdot 4)}{4^3} + \frac{(-1)^{5+1}(12 \cdot 5)}{5^3} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 12n}{n^3} = \boxed{\sum_{n=1}^{\infty} \frac{12(-1)^{n+1} n}{n^3}}
\end{aligned}$$

Lesson 17: Geometric Series and Convergence (II)

1. Solutions to In-Class Examples

Example 1. A ball has the property that each time it falls from a height h onto the ground, it will rebound to a height of rh for some $0 < r < 1$. Find the total distance traveled by the ball if $r = \dfrac{1}{3}$ and it is dropped from a height of 9 feet.

Solution: We draw a picture to get a feel for what is going on.



Notice that other than when we originally drop the ball, at each step the distance traveled by the ball is doubled because we must include the height the ball rebounds to and the distance the ball travels as it falls to the ground. The heights the ball travels are are

$$9,\ 3,\ 3,\ 1,\ 1,\ \frac{1}{3},\ \frac{1}{3},\cdots.$$

Observe

$$3 = (9)\frac{1}{3} = (9)\left(\frac{1}{3}\right)^1$$

$$1 = (3)\frac{1}{3} = \underbrace{\left((9)\frac{1}{3}\right)}_{3}\left(\frac{1}{3}\right) = 9\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = (9)\left(\frac{1}{3}\right)^2.$$

From this we can determine a pattern: the distance the ball travels is described by

$$9 + 2\sum_{n=1}^{\infty}(9)\left(\frac{1}{3}\right)^n = 9 + 18\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n.$$

This is clearly a geometric series so we use the geometric series formula to compute this sum. But our series starts at $n = 1$ (**not** $n = 0$), so we can't apply our formula just yet. Instead, write

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+1} = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n.$$

Hence,

$$\begin{aligned} 9 + 18\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n &= 9 + 18\left(\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\right) \\ &= 9 + \frac{18}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n \\ &= 9 + 6\left(\frac{1}{1 - 1/3}\right) \\ &= 9 + 6\left(\frac{1}{2/3}\right) \\ &= 9 + 6\left(\frac{3}{2}\right) \\ &= 9 + \frac{18}{2} \\ &= 9 + 9 = \boxed{18 \text{ feet}}. \end{aligned}$$

EXAMPLE 2. Suppose that in a country, 75% of all income the people receive is spent and 25% is saved. What is the total amount of spending generated in the long run by a \$10 billion tax rebate which is given to the country's citizens to stimulate the economy if saving habits do not change? Include the government rebate as part of the total spending.

Solution: The question is asking us to determine what is spent from now to the end of time (assuming the pattern holds). Since we are including the government rebate as part of the spending, we see at time $n = 0$, \$10 billion is spent. But, according to what they tell us, the citizens then spend 75% of the \$10 billion. So at time $n = 1$, \$10(.75) billion is spent. At time $n = 2$, the citizens spend $\$10(.75)(.75) = \$\,10(.75)^2$ billion and we continue on in this way. Again, we assume the pattern holds indefinitely.

Our goal is to find the **total** amount spent (measured in billions), which is the sum of all that is spent over time $n = 0, 1, 2, ...$ This is described by the summation

$$\underbrace{10}_{n=0} + \underbrace{10(.75)}_{n=1} + \underbrace{10(.75)^2}_{n=2} + \cdots = \sum_{n=0}^{\infty}10(.75)^n = 10\sum_{n=0}^{\infty}(.75)^n.$$

Because $n = 0$ and $.75 < 1$, we can apply our formula for the geometric series to determine that the total amount spent (in billions) is

$$10\sum_{n=0}^{\infty}(.75)^n = 10\left(\frac{1}{1-.75}\right) = 10\left(\frac{1}{.25}\right) = 10(4) = \boxed{40\text{ billion}}.$$

EXAMPLE 3. How much should you invest today at an annual interest rate of 4% compounded continuously so that in 3 years from today, you can make annual withdrawals of \$2000 in perpetuity? Round your answer to the nearest cent.

Solution: The question is asking: what do we need to invest today so that every year, we have \$2000 in the bank. The formula for continuously compounded annual interest is

$$A = Pe^{rt}$$

where r is the interest rate, t is time in years, A is the amount we have in the bank after t years, and P is the investment we make today. Let P_3 be the amount we invest today so that in 3 years, we have \$2000. Then, at the interest rate we are given,

$$2000 = P_3e^{.04(3)} \quad\Rightarrow\quad P_3 = 2000e^{-.04(3)}.$$

Let P_4 be the amount we invest today so that in 4 years, we have \$2000. Write

$$2000 = P_4e^{.04(4)} \quad\Rightarrow\quad P_4 = 2000e^{-.04(4)}.$$

Similarly, for any year $n > 3$ we can let P_n be the amount we invest today so that after n years, we have \$2000. Then

$$2000 = P_ne^{.04(n)} \quad\Rightarrow\quad P_n = 2000e^{-.04(n)}.$$

Where does this leave us? Well, the sum of all these P_n gives the total amount we need to invest today so that we will always have \$2000 in the bank each year beginning 3 years from now. So

$$\text{Total} = P_3 + P_4 + P_5 + \cdots = \sum_{n=3}^{\infty}2000e^{-.04(n)}.$$

$\sum_{n=3}^{\infty}2000e^{-.04(n)}$ is clearly a geometric series, but it is not in the correct form. We will need to use the formula for the geometric series but our series is not in the correct form. Write

$$\begin{aligned}\sum_{n=3}^{\infty}2000e^{-.04(n)} &= \sum_{n=3}^{\infty}2000\left(e^{-.04}\right)^n \\ &= \sum_{n=0}^{\infty}2000\left(e^{-.04}\right)^{n+3} \\ &= \sum_{n=0}^{\infty}2000\left(e^{-.04}\right)^3\left(e^{-.04}\right)^n\end{aligned}$$

$$= 2000e^{-.04(3)} \sum_{n=0}^{\infty} \left(e^{-.04}\right)^n.$$

Now that this is in the correct form and $|e^{-.04}| < 1$, we can apply the geometric formula.

The total we invest today is

$$2000e^{-.04(3)} \sum_{n=0}^{\infty} \left(e^{-.04}\right)^n = 2000e^{-.04(3)} \left(\frac{1}{1 - e^{-.04}}\right) \approx \boxed{\$45{,}238.85}.$$

EXAMPLE 4. 500 people are sent to a colony on Mars and each subsequent year 500 more people are added to the population of the colony. The annual death proportion is 5%. Find the eventual population of the Mars colony after many years have passed, just before a new group of 500 people arrive.

Solution: Let P_k be the population of the colony on Mars at the start of year k. Then $P_0 = 500$ because 500 people were sent to Mars initially. Moreover,

$$P_1 = \underbrace{500}_{\substack{\text{people sent}\\ \text{to Mars}}} + \underbrace{(P_0 - .05P_0)}_{\substack{\text{population already}\\ \text{on Mars}}}.$$

Similarly,

$$P_2 = \underbrace{500}_{\substack{\text{people sent}\\ \text{to Mars}}} + \underbrace{(P_1 - .05P_1)}_{\substack{\text{population already}\\ \text{on Mars}}}$$

and we continue on in this pattern. But we want a nicer way to write this. Try

$$P_1 = 500 + (P_0 - .05P_0) = 500 + .95P_0 = 500 + .95(\underbrace{500}_{P_0})$$

and

$$P_2 = 500+(P_1-.05P_1) = 500+.95P_1 = 500+.95(\underbrace{500 + .95(500)}_{P_1}) = 500+.95(500)+(.95)^2(500).$$

So our pattern is given by

$$\sum_{n=0}^{\infty} 500(.95)^n.$$

This is in the correct form to apply the geometric series formula. So we can write

$$\sum_{n=0}^{\infty} 500(.95)^n = \frac{500}{1 - .95} = \frac{500}{.05} = 10{,}000.$$

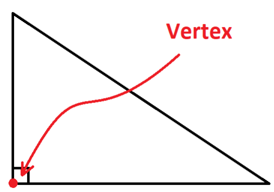
We aren't quite done though. We were asked to find the population **just before** a new group of 500 people arrive. <u>So we need to subtract 500</u>. Thus, our answer is $\boxed{9{,}500}$.

2. Additional Examples

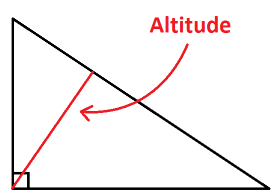
1. In a right triangle, a series of altitudes are drawn starting with an altitude drawn using the vertex of the right angle and drawn towards the hypotenuse. Then subsequently continuing to draw altitudes from the right angles in the new right triangles created, and which also include the vertex from the smallest angle of the original right triangle. The series of altitudes are drawn so they move closer and closer to the smallest angle in the original right triangle. Find the sum of the lengths of these altitudes given that one of the angles of the original triangle is 47° and the side of the triangle adjacent to this angle is 2.7. Round your answer to the nearest hundredth.

Solution: This is the most difficult problem in Math 16020. The biggest challenge is that, if you compute using the numbers given, it's very easy to oversimplify which makes you miss the overarching pattern. Instead of using the numbers given, we will use variables and then substitute what we are given at the very end.

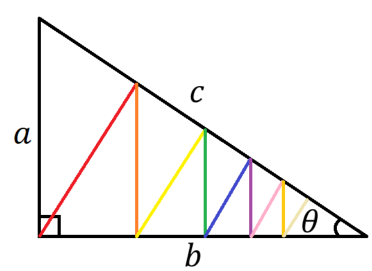
The first issue in this problem is understanding what object is being described. The vertex of a right triangle is the point where the smaller legs (by legs of a triangle, I mean the two shorter sides of a right triangle) meet to form the right angle:



The altitude from the vertex of a right triangle is the line starting from the vertex that makes a right angle with the hypotenuse:



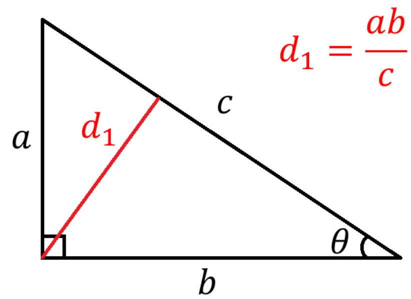
Now, we are drawing a series of altitudes in our triangles which always contains the **smallest** angle, which we will call θ. Consider the following picture:



We let this right triangle have sides a, b, c where a is opposite θ and c is the hypotenuse.

Interestingly, we have a very nice formula for the length of an altitude when compared to the sides of the triangle. For example, if d_1 is the length of the altitude in the triangle abc, then

$$d_1 = \frac{ab}{c}.$$



In general, the length of an altitude of this type is the product of the legs of the triangle divided by the hypotenuse.

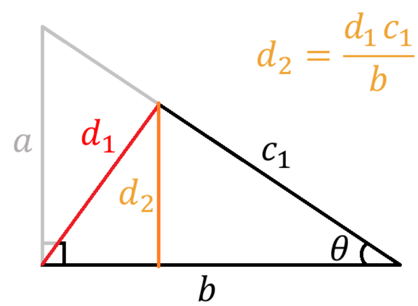
Observe that because

$$\cos\theta = \frac{b}{c},$$

we may conclude that

$$d_1 = \frac{ab}{c} = a\left(\frac{b}{c}\right) = a\cos\theta.$$

With this in mind, we look at the next altitude in our sequence:



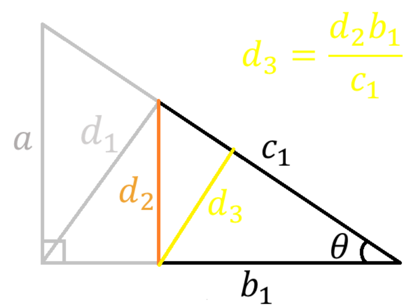
Note that since we are not looking at the original triangle anymore (because this new altitude isn't an altitude in the first triangle) we have to be careful about the lengths of our sides. Here, our new triangle has legs d_1, c_1, and hypotenuse b. Observe that b is part of the previous triangle and we have already computed the length of d_1, but we need to determine the length of c_1. Notice that

$$\cos\theta = \frac{c_1}{b} \quad \Rightarrow \quad b\cos\theta = c_1.$$

So,

$$d_2 = \frac{d_1 c_1}{b} = \frac{d_1(b\cos\theta)}{b} = d_1\cos\theta = \underbrace{(a\cos\theta)}_{d_1}\cos\theta = a(\cos\theta)^2.$$

Next,



In this triangle our altitude is d_3, our legs are d_2, b_1, and our hypotenuse is c_1. We need to compute the length of c_1. Since

$$\cos\theta = \frac{b_1}{c_1} \quad \Rightarrow \quad b_1 = c_1\cos\theta.$$

Then, we know that

$$d_3 = \frac{d_2 b_1}{c_1} = \frac{d_2 c_1\cos\theta}{c_1} = d_2\cos\theta = \underbrace{(a(\cos\theta)^2)}_{d_2}\cos\theta = a(\cos\theta)^3.$$

From this we can determine a pattern. We see that the sum of the lengths of these altitudes is given by

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} a(\cos\theta)^n = a\cos\theta + a(\cos\theta)^2 + a(\cos\theta)^3 + \cdots.$$

We put this in the correct form to apply the geometric series formula:

$$\begin{aligned}\sum_{n=1}^{\infty} a(\cos\theta)^n &= \sum_{n=0}^{\infty} a(\cos\theta)^{n+1} \\ &= \sum_{n=0}^{\infty} (a\cos\theta)(\cos\theta)^n \\ &= \frac{a\cos\theta}{1-\cos\theta}.\end{aligned}$$

Now that we have the general formula, we need to input the numbers they have given us.

After carefully re-reading the problem, we see that $a = 2.7$ and that $\theta = 90° - 47° = 43°$. Thus, the sum of the lengths of all the altitudes is

$$\frac{2.7\cos(43°)}{1-\cos(43°)} \approx \boxed{7.35}.$$

TL;DR: The geometric series describing this situation is

$$\sum_{n=1}^{\infty} a(\cos x)^{n+1} = a\cos x + a(\cos x)^2 + a(\cos x)^3 + \cdots$$

where a is the length of the side they give you and x is 90 minus the angle they give you (so if they give you the angle $60°$, $x = 30°$). Note that x must be measured in **degrees**. The formula for this sum is

$$\frac{a\cos x}{1-\cos x}.$$

2. Determine if the series converges; if so, find its sum:

$$\frac{2304}{25} - \frac{48}{5} + 1 - \frac{5}{48} + \frac{25}{2304} - \cdots.$$

Solution: Notice that $\left(-\frac{5}{48}\right)^0 = 1$ and that $\left(-\frac{5}{48}\right)^{-1} = -\frac{48}{5}$. Hence,

$$\begin{aligned}
&\frac{2304}{25} - \frac{48}{5} + 1 - \frac{5}{48} + \frac{25}{2304} - \cdots \\
&= \left(-\frac{5}{48}\right)^{-2} + \left(-\frac{5}{48}\right)^{-1} + \left(-\frac{5}{48}\right)^{0} + \left(-\frac{5}{48}\right)^{1} + \left(-\frac{5}{48}\right)^{2} + \cdots \\
&= \sum_{n=-2}^{\infty} \left(-\frac{5}{48}\right)^n
\end{aligned}$$

Since $|r| = \left|-\frac{5}{48}\right| < 1$, this series $\boxed{\text{converges}}$. To find its sum, we need to put our series into the correct form. Write

$$\begin{aligned}
\sum_{n=-2}^{\infty} \left(-\frac{5}{48}\right)^n &= \sum_{n=0}^{\infty} \left(-\frac{5}{48}\right)^{n-2} \\
&= \sum_{n=0}^{\infty} \left(-\frac{5}{48}\right)^{-2} \left(-\frac{5}{48}\right)^{n} \\
&= \frac{\left(-\frac{5}{48}\right)^{-2}}{1-\left(-\frac{5}{48}\right)}
\end{aligned}$$

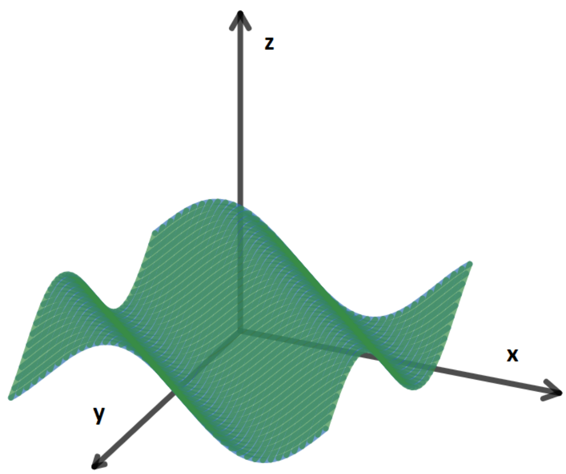
$$= \frac{\left(-\frac{5}{48}\right)^{-2}}{1 + \frac{5}{48}}$$

$$= \boxed{\frac{110{,}592}{1325}}$$

Lesson 18: Introduction to Functions of Several Variables

1. Functions of Several Variables

Today we introduce **functions of several variables**. For the purposes of this class, we primarily consider functions with 2 inputs, x and y. For example, how the weather feels to humans depends on both the temperature and the humidity — two variables that do not depend on each other. Geometrically, by adding another input, we are adding another dimension to our graphs which we label z.



We write $z = f(x, y)$, which means that z depends on x and y.

NOTE 48 (Caution). Before, we would write $y = f(x)$ which means that y is related to x. Now, we write $z = f(x, y)$, which means z is related to x and y **BUT** this does not mean x and y are related to each **other**. x and y are just the *inputs* and will act independently.

EXAMPLES.

1. If $f(x, y) = \dfrac{x}{\ln(2y)}$, find $f\left(1, \dfrac{e^3}{2}\right)$.

Solution: A good question here is to ask: which input is x and which is y? Fortunately, we will always write our multivariable functions as $z = f(x, y)$ so

$$z = f\left(\underset{\substack{\uparrow \\ x}}{\boxed{1}}, \underset{\substack{\uparrow \\ y}}{\boxed{\dfrac{e^3}{2}}}\right)$$

Thus,

$$f\left(1, \frac{e^3}{2}\right) = \frac{1}{\ln\left(\frac{2e^3}{2}\right)} = \frac{1}{\ln e^3} = \boxed{\frac{1}{3}}.$$

2. Find the domain of

$$f(x, y) = \frac{\sqrt{11x - 3y + 2}}{x}.$$

Solution: Just as with single variable functions, functions of several variables can have issues of domain — which is why we ask these sorts of questions. Fortunately, nothing really changes from the single variable case.

Finding the Domain: We have to check the following three things to make sure our function is defined.

(1) No dividing by zero

Ex: $\dfrac{1}{x+y}$ doesn't exist when $x + y = 0$

(2) Even roots have non-negative input

Ex: $\sqrt{1 + x + y}$ has issues whenever $x + y < -1$ because then the input is negative. But $\sqrt[3]{1 + x + y}$ has no issues whatsoever.

(3) ln has positive input

Ex: $\ln(x + 2y)$ doesn't exist when $x + 2y \leq 0$

Sometimes these 3 things can overlap, for example,

$$f(x, y) = \frac{1}{\ln(x + y)}$$

requires you to check (1) and (3). To not divide by zero, we can't have $\ln(x + y) = 0 \Rightarrow x + y = 1$. To make sure $\ln(x + y)$ exists, we must have $x + y > 0$. This means our domain is

$$\{(x, y) : x + y > 0 \text{ and } x + y \neq 1\}.$$

Returning to $f(x, y) = \dfrac{\sqrt{11x - 3y + 2}}{x}$, we check (1) and (2). That is, we must have $x \neq 0$ and $11x - 3y + 2 \geq 0 \;\Rightarrow\; 11x - 3y \geq -2$. Our domain is then

$$\boxed{\{(x, y) : x \neq 0, 11x - 3y \geq -2\}}.$$

3. Find the domain of

$$f(x, y) = \frac{\sqrt{x - 1}}{\ln(y - 2) - 3}.$$

Solution: We need to check (1), (2), and (3).

(1) If $\ln(y-2)-3=0$, then our function does not exist. This means that

$$\ln(y-2)=3 \;\Rightarrow\; y-2=e^3 \;\Rightarrow\; y=e^3+2.$$

So we must exclude $y=e^3+2$.

(2) We have an even root, so we need $x-1\geq 0 \;\Rightarrow\; x\geq 1$.

(3) For $\ln(y-2)$ to exist, we must have $y-2>0 \;\Rightarrow\; y>2$.

Putting this all together, our domain is

$$\boxed{\{(x,y): y\neq e^3+2, x\geq 1, y>2\}}.$$

4. Find the range of

$$f(x,y)=3\sqrt{y+5x^2}.$$

Solution: Although this is a function of several variables, the output is a single real number. Take the input, $y+5x^2$ and replace it by t, that is, write

$$3\sqrt{y+5x^2}=3\sqrt{t}.$$

Thus, the output (which is the range) should match the range of $3\sqrt{t}$. The z-values achieved by $3\sqrt{t}$ are $[0,\infty)$.

Finding the Range: The range is the collection of z-values the function achieves.
To find the range, replace the input by t, and write down the range of the resulting function.

Example 1. Find the range of $f(x,y)=\ln(x^2-y)$.
The input is x^2-y, so we consider $\ln(t)$. The range of $\ln(t)$ is $(-\infty,\infty)$. So the f has range of **all real numbers**, that is $-\infty<z<\infty$ or $(-\infty,\infty)$.

2. Level Curves

Now, we talk about how we work to understand functions of several variables. These functions are much more difficult to graph so we need to understand them using different techniques, one such method being **level curves**. The idea is to choose a point on the z-axis then take a "slice" of the function to see what it's doing at that height.

Ex 1. Let $f(x,y)=\ln(x^2+y^2)$ and suppose we want to see what is going on with this function. Without access to some graphing instrument, this will be very tricky to draw. Observe that if we choose $z=C$ for some arbitrary, but constant, C, then

$$C=\underbrace{\ln(x^2+y^2)}_{f(x,y)} \iff e^C=x^2+y^2.$$

For each C, we are looking at **circle centered at the origin of radius** $\sqrt{e^C}$. We would choose some value for C, like $C = \ln 4$, then write

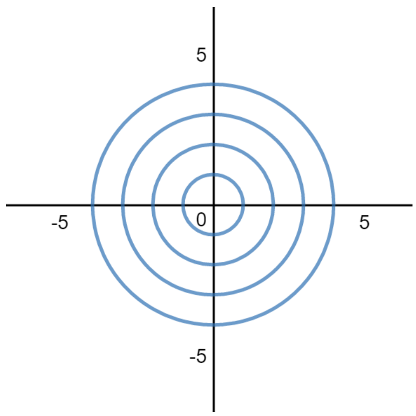
$$\underbrace{e^{\ln 4}}_{4} = x^2 + y^2$$

which gives us a picture of a circle centered at $(0, 0)$ of radius 2. This is the level curve of f at $C = \ln 4$.

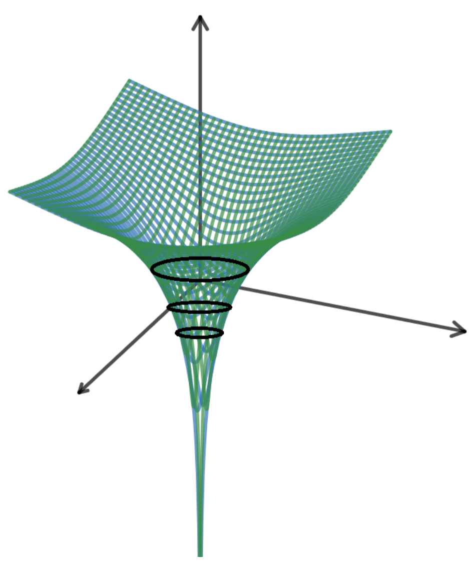
If we look at the level curves associated to

$$C = 0, \ln 4, \ln 9, \ln 16,$$

we get the following picture:



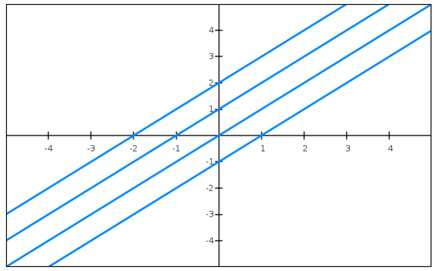
These are level curves from which we can construct the actual graph of $f(x, y)$. Think of level curves as what happens if we take a 3-dimensional image and smash it flat on the floor (so the level curves are a bird's eye view of the graph). To get the graph of $f(x, y)$, we use C as labels for the height of the function:



Observe that all the cross-sections of this graph are circles of increasing radius, which we see for our different z-values.

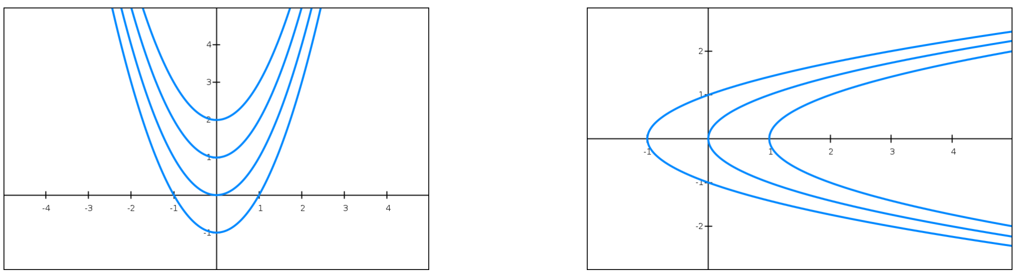
NOTE 49. Recall that $(x - h)^2 + (y - k)^2 = r^2$ is a circle of radius r centered at (h, k).

Level curves can also come in the following shapes:



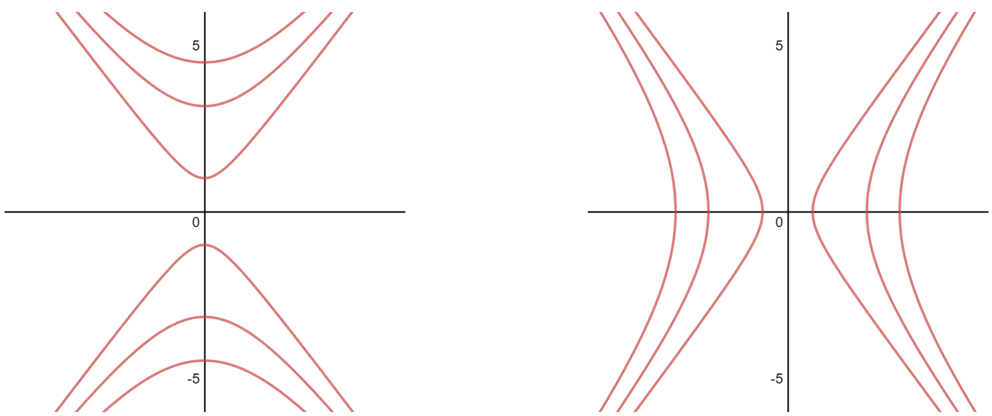
$$x + y = z$$

FIGURE 5. Lines



$x^2 - y = z$ $\qquad$ $y^2 - x = z$

FIGURE 6. Parabolas



$y^2 - x^2 = z$ $\qquad$ $x^2 - y^2 = z$

FIGURE 7. Hyperbolas

This list isn't exhaustive as level curves can appear as any function.

EXAMPLES.

5. Consider the function $f(x, y) = x^2 y$.

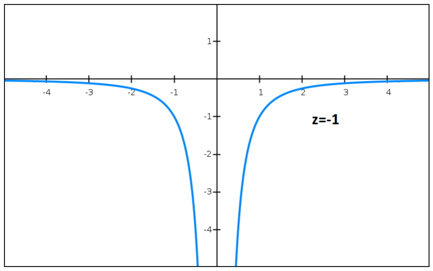
(a) Find the level curves of $f(x, y)$ for $z = -1, z = 2$.

Solution: Because we have been given specific z-values, we **don't** need to consider any other values. So, really, we are looking at the functions

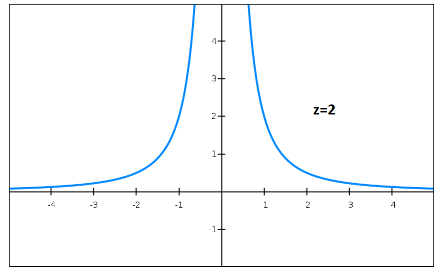
$$-1 = x^2 y \quad \text{and} \quad 2 = x^2 y.$$

We want to graph these functions.

$y = -\dfrac{1}{x^2} \longleftarrow$ rational function



$y = \dfrac{2}{x^2} \longleftarrow$ rational function



We observe that the symmetry is about the y-**axis**.

(b) What are the vertical and horizontal asymptotes for these functions?

Solution:

Horizontal Asymptote: $\boxed{y = 0}$

Vertical Asymptote: $\boxed{x = 0}$

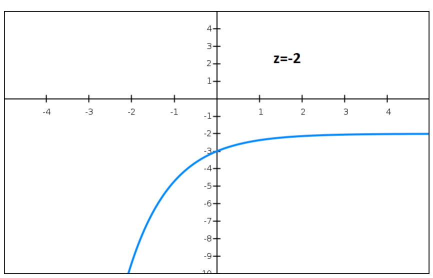
6. Consider the function $f(x, y) = e^{-x} + y$.

(a) Find the level curves of $f(x, y)$ for $z = -2, 1$.

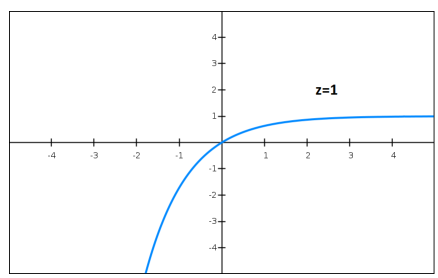
Solution: Again, we need only focus on $z = -2, 1$. This means we are looking at the functions

$$-2 = e^{-x} + y \quad \text{and} \quad 1 = e^{-x} + y.$$

$y = -e^{-x} - 2 \longleftarrow$ exponential function



$y = -e^{-x} + 1 \longleftarrow$ exponential function



(b) Find the horizontal asymptotes of these functions.

Solution: We have $\boxed{y = -2}$ and $\boxed{y = 1}$.

(c) Find the y-intercepts of these functions.

Solution: The y-intercept is just where $x = 0$. So

$$y = -e^0 - 2 \;\Rightarrow\; \boxed{y = -3}$$

and

$$y = -e^0 + 1 \;\Rightarrow\; \boxed{y = 0}.$$

3. Additional Examples

EXAMPLES.

1. Find half of the area of the domain of

$$f(x,y) = \frac{8x}{\sqrt{16 - x^2 - y^2}}$$

in the xy-plane.

Solution: We first determine the domain of $f(x, y)$. Since there is no natural log in the function, we check only (1) and (2).

(1) We need $\sqrt{16 - x^2 - y^2} \neq 0$. This means that $16 - x^2 - y^2 \neq 0$ and so we need to avoid $x^2 + y^2 = 16$, which is a circle of radius 4 centered at the origin.

(2) For a square root to make sense, the input must be non-negative. So, we have

$$16 - x^2 - y^2 \geq 0 \;\Rightarrow\; 16 \geq x^2 + y^2.$$

Combining items (1) and (2), we must have

$$x^2 + y^2 < 16.$$

Thus, the domain is the interior of the circle of radius 4 centered at the origin.

Second, we find the half of the area of the domain. If the domain is a circle of radius 4, then half of this area is

$$\frac{\pi(4)^2}{2} = \boxed{8\pi}.$$

2. Find the level curves of

$$f(x,y) = 3\sqrt{y + 5x^2}.$$

Solution: We write

$$C = 3\sqrt{y + 5x^2}$$

and solve for something we can graph. So,

$$\begin{aligned} C &= 3\sqrt{y + 5x^2} \\ \Rightarrow \quad \frac{C}{3} &= \sqrt{y + 5x^2} \\ \Rightarrow \quad \frac{C^2}{9} &= y + 5x^2 \\ \Rightarrow \quad \frac{C^2}{9} - 5x^2 &= y \end{aligned}$$

Thus, we see each level curve is a $\boxed{\text{downward opening parabola shifted by } \frac{C^2}{9}}$.

3. If $f(x,y) = \sqrt{11(x+3)^3 + 11(y+19)^2}$ and $C = 1$, then describe the level curves of $f(x,y) = C$.

Solution: We are told that

$$\sqrt{11(x+3)^3 + 11(y+19)^2} = 1.$$

We make some simplifications to get a better idea of what the level curves look like. Write

$$\begin{aligned} 1 &= \sqrt{11(x+3)^2 + 11(y+19)^2} \\ \Rightarrow \quad 1^2 &= 11(x+3)^2 + 11(y+19)^2 \\ \Rightarrow \quad \frac{1}{11} &= (x+3)^2 + (y+19)^2 \end{aligned}$$

By note (49), our equation is a $\boxed{\text{circle}}$ of radius $\boxed{\sqrt{\frac{1}{11}}}$ centered at $\boxed{(-3,-19)}$.

Lesson 19: Partial Derivatives

1. Partial Derivatives

We address how to take a derivative of a function of several variables. Although we won't get into the details, the idea is that we take a derivative with respect to a "direction". What we mean is this: if we have a function of several variables, we choose 1 variable and take the derivatives thinking of all the other variables as **constants**. But this type of derivative doesn't give the entire picture of what the function is doing so we call these **partial derivatives**.

Ex 1. Let $f(x, y) = x + 2y$ and suppose we want to find its partial derivatives.

First, we need to choose a variable, say x. Second, we think of the other variables (in this case just y) as **constant with respect to** x. This means we think of x and y as acting totally independently so x changing doesn't affect y. We use a special notation denote this concept: $\frac{\partial}{\partial x}$.

The partial derivative **with respect to** (wrt) x is:

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x}(x + 2y) = \frac{\partial}{\partial x}(x) + \underbrace{\frac{\partial}{\partial x}(2y)}_{\substack{y \text{ does not} \\ \text{change wrt to } x}} = 1 + 0 = 1.$$

Notice that $\frac{\partial}{\partial x}(x) = \frac{d}{dx}(x)$. This is because with respect to x, $\frac{\partial}{\partial x}$ is *exactly* the derivative as we've always done it.

We use the same line of thinking when we take y and hold x fixed. Here, x is a constant with respect to y. Again, we have our own notation: $\frac{\partial}{\partial y}$.

The partial derivative with respect to y is:

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y}(x + 2y) = \underbrace{\frac{\partial}{\partial y}(x)}_{\substack{x \text{ does not} \\ \text{change wrt } y}} + \frac{\partial}{\partial y}(2y) = 0 + 2 = 2.$$

Again, we see that $\frac{\partial}{\partial y}(2y) = \frac{d}{dy}(2y)$ because with respect to y, $\frac{\partial}{\partial y}$ is the same y derivative as before.

Remark 50. This ∂ is not a d, and we will call it "del". ∂ is used exclusively for partial derivatives.

NOTE 51. We will use a variety of notation for partial derivatives but they will mean the same thing. For example, if our function is $z = f(x, y)$, we can write

$$f_x = f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$$

and

$$f_y = f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}.$$

For $f(x, y) = x + 2y$, our partial derivatives are

$$\frac{\partial f}{\partial x} = 1 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2.$$

Be sure to review all the differentiation rules you may have forgotten (Appendix B).

EXAMPLES.

1. Find f_x, f_y if $f(x, y) = e^{x^2} + \ln y^2$.

Solution:

$$f_x(x, y) = \frac{\partial}{\partial x}(e^{x^2} + \ln y^2) = \frac{\partial}{\partial x}(e^{x^2}) + \underbrace{\frac{\partial}{\partial x}(\ln y^2)}_{0} = 2xe^{x^2}$$

$$f_y(x, y) = \frac{\partial}{\partial y}(e^{x^2} + \ln y^2) = \underbrace{\frac{\partial}{\partial y}(e^{x^2})}_{0} + \frac{\partial}{\partial y}(\ln y^2) = \frac{2y}{y^2} = \frac{2}{y}$$

Thus,

$$\boxed{f_x = 2xe^{x^2} \quad \text{and} \quad f_y = \frac{2}{y}}$$

2. Find f_x, f_y if $f(x, y) = y \cos x$.

Solution: For this problem, we want to remember that whenever c is a constant

$$\frac{d}{dx}(cx^3) = c\frac{d}{dx}(x^3).$$

We have the same property for partial derivatives:

$$\frac{\partial}{\partial x}(cx^3) = c\frac{\partial}{\partial x}(x^3).$$

Even more, any **function of** y is constant with respect to x. So

$$\frac{\partial}{\partial x}((\sin y)x^3) = \sin(y)\frac{\partial}{\partial x}(x^3).$$

With this in mind, we compute

$$f_x(x,y) = \frac{\partial}{\partial x}(y\cos x) = \underset{\substack{\uparrow \\ \text{constant} \\ \text{wrt } x}}{y} \frac{\partial}{\partial x}(\cos x) = -y\sin x$$

$$f_y(x,y) = \frac{\partial}{\partial y}(y\cos x) = \underset{\substack{\uparrow \\ \text{constant} \\ \text{wrt } y}}{\cos x} \underbrace{\frac{\partial}{\partial y}(y)}_{1} = \cos x$$

Thus,

$$\boxed{f_x = -y\sin x \quad \text{and} \quad f_y = \cos x}.$$

3. Find $f_x(1,0)$ and $f_y(1,0)$ if $f(x,y) = \dfrac{3x-y}{1-y}$.

Solution: The difference between this example and the examples above is that here we need to differentiate *and then* evaluate the derivative at the point $(1,0)$.

Differentiating with respect to x,

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{3x-y}{1-y}\right) = \underbrace{\frac{1}{1-y}}_{\substack{\text{constant} \\ \text{wrt } x}} \frac{\partial}{\partial x}(3x-y) = \frac{1}{1-y}(3-0) = \frac{3}{1-y}.$$

Hence,

$$f_x(1,0) = \frac{3}{1-0} = 3.$$

Now, to differentiate with respect to y, we will need to use the quotient rule (we could also use the product rule after a small rewrite). So

$$\begin{aligned}
f_y(x,y) &= \frac{\partial}{\partial y}\left(\frac{3x-y}{1-y}\right) \\
&= \frac{(1-y)\frac{\partial}{\partial y}(3x-y) - (3x-y)\frac{\partial}{\partial y}(1-y)}{(1-y)^2} \\
&= \frac{(1-y)(-1) - (3x-y)(-1)}{(1-y)^2} \\
&= \frac{(-1+y) + (3x-y)}{(1-y)^2} \\
&= \frac{y - 1 + 3x - y}{(1-y)^2} \\
&= \frac{3x-1}{(1-y)^2}.
\end{aligned}$$

Thus,

$$f_y(1,0) = \frac{3(1)-1}{(1-0)^2} = 2.$$

Finally,

$$\boxed{f_x(1,0) = 3 \quad \text{and} \quad f_y(1,0) = 2}.$$

4. Find f_x, f_y if $f(x,y) = e^{x^2 y}$.

Solution: Recall that if we were considering a function of a single variable, say e^{3x+x^2}, its derivative with respect to x is

$$\frac{d}{dx}e^{3x+x^2} = \left(\frac{d}{dx}(3x+x^2)\right)e^{3x+x^2} = (3+2x)e^{3x+x^2}$$

by the chain rule. The chain rule still applies to partial derivatives.

$$f_x = \frac{\partial}{\partial x}(e^{x^2 y}) = \frac{\partial}{\partial x}(x^2 y)e^{x^2 y} = \underset{\substack{\uparrow \\ \text{constant} \\ \text{wrt } x}}{y} \frac{\partial}{\partial x}(x^2)e^{x^2 y} = y(2x)e^{x^2 y} = 2xye^{x^2 y}$$

$$f_y = \frac{\partial}{\partial y}(e^{x^2 y}) = \frac{\partial}{\partial y}(x^2 y)e^{x^2 y} = \underset{\substack{\uparrow \\ \text{constant} \\ \text{wrt } y}}{x^2} \frac{\partial}{\partial y}(y)e^{x^2 y} = x^2 e^{x^2 y}$$

So,

$$\boxed{f_x = 2xye^{x^2 y} \quad \text{and} \quad f_y = x^2 e^{x^2 y}}.$$

5. Find $f_x(1,0)$ if $f(x,y) = \ln(\ln(y)x)$.

Solution: Recall that

$$\frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)}.$$

This rule still follows for partial derivatives:

$$f_x(x,y) = \frac{\partial}{\partial x}(\ln(\ln(y)x)) = \frac{\frac{\partial}{\partial x}(\ln(y)x)}{\ln(y)x} = \frac{\ln(y)\frac{\partial}{\partial x}(x)}{\ln(y)x} = \frac{\ln(y)}{\ln(y)x} = \frac{1}{x}$$

Thus,

$$f_x(1,0) = \frac{1}{1} = \boxed{1}.$$

2. Additional Examples

Examples.

1. Find f_x and f_y given

$$f(x,y)=\sqrt{1-7x^2-3y^2}.$$

Solution: Our function will require us to use the chain rule. Observe that

$$\sqrt{1-7x^2-3y^2}=(1-7x^2-3y^2)^{1/2}.$$

Differentiating first with respect to x,

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}(1-7x^2-3y^2)^{1/2} \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x}(1-7x^2-3y^2)\right)(1-7x^2-3y^2)^{-1/2} \\ &= \frac{1}{2}(-14x)(1-7x^2-3y^2)^{-1/2} \\ &= \frac{-7x}{\sqrt{1-7x^2-3y^2}} \end{aligned}$$

Next, we differentiate with respect to y,

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}(1-7x^2-3y^2)^{1/2} \\ &= \frac{1}{2}\left(\frac{\partial}{\partial y}(1-7x^2-3y^2)\right)(1-7x^2-3y^2)^{-1/2} \\ &= \frac{1}{2}(-6y)(1-7x^2-3y^2)^{-1/2} \\ &= \frac{-3y}{\sqrt{1-7x^2-3y^2}} \end{aligned}$$

Thus,

$$\boxed{f_x=\frac{-7x}{\sqrt{1-7x^2-3y^2}} \quad \text{and} \quad f_y=\frac{-3y}{\sqrt{1-7x^2-3y^2}}}.$$

2. Let

$$f(x,y)=\frac{10x^2y^3}{y-8x};$$

evaluate $f_x(x,y)$ at $(1,-1)$. Round your answer to 4 decimal places.

Solution: We differentiate with respect to x and then evaluate at the point $(x,y)=(1,-1)$. Note that

$$\frac{10x^2y^3}{y-8x}=10x^2y^3(y-8x)^{-1}.$$

Written this way, we can use the product rule instead of the quotient rule, which is how we proceed.

Write

$$
\begin{aligned}
f_x &= \frac{\partial}{\partial x}(10x^2y^3(y-8x)^{-1}) \\
&= 10x^2y^3\left(\frac{\partial}{\partial x}(y-8x)^{-1}\right) + \left(\frac{\partial}{\partial x}(10x^2y^3)\right)(y-8x)^{-1} \\
&= 10x^2y^3\left(-\frac{\partial}{\partial x}(y-8x)\right)(y-8x)^{-2} + 20xy^3(y-8x)^{-1} \\
&= -10x^2y^3(-8)(y-8x)^{-2} + 20xy^3(y-8x)^{-1} \\
&= 80x^2y^3(y-8x)^{-2} + 20xy^3(y-8x)^{-1} \\
&= (y-8x)^{-2}\left(80x^2y^3 + 20xy^3(y-8x)\right) \\
&= (y-8x)^{-2}\left(80x^2y^3 + 20xy^4 - 160x^2y^3\right) \\
&= (y-8x)^{-2}\left(20xy^4 - 80x^2y^3\right)
\end{aligned}
$$

Evaluating at $(1, -1)$, we have

$$
\begin{aligned}
f_x(1,-1) &= (-1-8(1))^{-2}(20(1)(-1)^4 - 80(1)^2(-1)^3) \\
&= (-9)^{-2}(20+80) \\
&\approx \boxed{1.2346}
\end{aligned}
$$

$\boxed{3.}$ Find f_x, f_y if $f(x,y) = xy\sin(xy)$.

Solution: We need to use the product rule and chain rule.

$$
\begin{aligned}
f_x(x,y) &= \frac{\partial}{\partial x}(xy\sin(xy)) \\
&= xy\frac{\partial}{\partial x}(\sin(xy)) + \frac{\partial}{\partial x}(xy)\sin(xy) \\
&= xy(\underset{\frac{\partial}{\partial x}(xy)}{\underset{\uparrow}{y}} \cdot \cos(xy)) + y\sin(xy) \\
&= xy^2\cos(xy) + y\sin(xy) \\
f_y(x,y) &= xy\frac{\partial}{\partial y}(\sin(xy)) + \frac{\partial}{\partial y}(xy)\sin(xy) \\
&= xy(\underset{\frac{\partial}{\partial y}(xy)}{\underset{\uparrow}{x}} \cdot \cos(xy)) + x\sin(xy) \\
&= x^2y\cos(xy) + x\sin(xy)
\end{aligned}
$$

Thus,

$$\boxed{f_x = xy^2\cos(xy) + y\sin(xy) \quad \text{and} \quad f_y = x^2y\cos(xy) + x\sin(xy)}.$$

Lesson 20: Partial Derivatives (II)

1. Second Order Partial Derivatives

Just as with functions of a single variable, it makes sense to take higher derivatives of functions of several variables.

We can take derivatives with respect to the same variable twice, which we would denote

$$f_{xx} = \frac{\partial^2 f}{(\partial x)^2} \quad \text{and} \quad f_{yy} = \frac{\partial^2 f}{(\partial y)^2}.$$

But we can also take the derivative with respect to one variable and *then* with respect to another. For example, we might take the derivative with respect to x and *then* with respect to y. We denote this by

$$(f_x)_y = f_{xy}.$$

Similarly, if we differentiate with respect to y and *then* with respect to x, we would write

$$(f_y)_x = f_{yx}.$$

FACT 52 (Clairaut's Theorem). $f_{xy} = f_{yx}$

So it turns out the distinction doesn't actually matter so much. However, there are situations where it is easier to differentiate with respect to one variable first and the other second.

EX 1. If $f(x, y) = y \sin(x) \cos(x)$ and we want to find f_{xy}, it is actually easier to differentiate with respect to y and then with respect to x.

We call f_{xy} and f_{yx} the **mixed partials**.

EXAMPLES.

1. Find the second order derivatives of

$$f(x, y) = x^3y^2 + xy^6.$$

Solution: When we are asked to find the second order derivatives, this means we need to find f_{xx}, f_{yy}, f_{xy}, i.e., *all* the second order derivatives.

We start by finding the first order derivatives:

$$f_x(x, y) = \frac{\partial}{\partial x}(x^3y^2 + xy^6) = 3x^2y^2 + y^6$$

$$f_y(x, y) = \frac{\partial}{\partial y}(x^3y^2 + xy^6) = 2x^3y + 6xy^5$$

Then we note that

$$f_{xx} = (f_x)_x, \quad f_{xy} = (f_x)_y, \quad f_{yy} = (f_y)_y.$$

So, starting from f_x, f_y, we find

$$f_{xx}(x,y) = \frac{\partial}{\partial x}(\underbrace{3x^2y^2 + y^6}_{f_x}) = 6xy^2$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(\underbrace{3x^2y^2 + y^6}_{f_x}) = 6x^2y + 6y^5$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}(\underbrace{2x^3y + 6xy^5}_{f_y}) = 2x^3 + 30xy^4$$

Therefore, our second order derivatives are

$$\boxed{f_{xx} = 6xy^2, \quad f_{yy} = 2x^3 + 30xy^4, \quad f_{xy} = 6x^2y + 6y^5}$$

2. Find f_{uv} if $f(u,v) = e^{7u+v}$.

Solution: Nothing is different here except how they have named the variables. Further, we need only find one second order derivative. Differentiating with respect to u, we get

$$f_u(u,v) = \frac{\partial}{\partial u}(e^{7u+v}) = \left[\frac{\partial}{\partial u}(7u+v)\right] \cdot e^{7u+v} = 7e^{7u+v}.$$

Then differentiating with respect to v, we get

$$f_{uv}(u,v) = \frac{\partial}{\partial v}(7e^{7u+v}) = 7\underbrace{\left[\frac{\partial}{\partial v}(7u+v)\right]}_{1} \cdot e^{7u+v} = \boxed{7e^{7u+v}}.$$

3. Find the second order derivatives of

$$f(x,y) = x\ln(3xy).$$

Solution: Again, when asked to find the second order derivatives, we are asked to find f_{xx}, f_{xy}, f_{yy}.

To start,

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}(x\ln(3xy)) \\ &= x\left[\frac{\partial}{\partial x}(\ln(3xy))\right] + \ln(3xy)\underbrace{\left[\frac{\partial}{\partial x}(x)\right]}_{1} \\ &= x\left(\frac{\frac{\partial}{\partial x}(3xy)}{3xy}\right) + \ln(3xy) \end{aligned}$$

$$= x\left(\frac{3y}{3xy}\right) + \ln(3xy)$$
$$= 1 + \ln(3xy)$$
$$f_y(x,y) = \frac{\partial}{\partial y}(x\ln(3xy))$$
$$= x\left[\frac{\partial}{\partial y}(\ln(3xy))\right]$$
$$= x\left(\frac{\frac{\partial}{\partial y}(3xy)}{3xy}\right)$$
$$= x\left(\frac{3x}{3xy}\right) = \frac{x}{y}$$

Next,

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\underbrace{(1+\ln(3xy))}_{f_x} = \frac{\frac{\partial}{\partial x}(3xy)}{3xy} = \frac{3y}{3xy} = \frac{1}{x}$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\underbrace{(1+\ln(3xy))}_{f_x} = \frac{\frac{\partial}{\partial y}(3xy)}{3xy} = \frac{3x}{3xy} = \frac{1}{y}$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\underbrace{\left(\frac{x}{y}\right)}_{f_y} = x\left[\frac{\partial}{\partial y}\left(\frac{1}{y}\right)\right] = x\left(-\frac{1}{y^2}\right) = -\frac{x}{y^2}$$

Thus,

$$\boxed{f_{xx} = \frac{1}{x}, \quad f_{yy} = -\frac{x}{y^2}, \quad f_{xy} = \frac{1}{y}}$$

4. If $f(x,y) = 8x\sin(8y)$, find $f_{xx}(5,10), f_{xy}(5,10), f_{yy}(5,10)$. Round your answers to the nearest hundredth.

Solution: We begin by computing the first order derivatives. Write

$$f_x = \frac{\partial}{\partial x}(8x\sin(8y))$$
$$= \sin(8y)\left[\frac{\partial}{\partial x}(8x)\right]$$
$$= 8\sin(8y)$$

$$f_y = \frac{\partial}{\partial y}(8x\sin(8y))$$
$$= 8x\left[\frac{\partial}{\partial y}(\sin(8y))\right]$$
$$= 8x\,[8\cos(8y)]$$
$$= 64x\cos(8y)$$

Next, we find the second order partial derivatives:

$$f_{xx} = \frac{\partial}{\partial x}(\underbrace{8\sin(8y)}_{f_x}) = 0$$

$$f_{xy} = \frac{\partial}{\partial y}(\underbrace{8\sin(8y)}_{f_x}) = 64\cos(8y)$$

$$f_{yy} = \frac{\partial}{\partial y}(\underbrace{64x\cos(8y)}_{f_y})$$
$$= 64x\left[\frac{\partial}{\partial y}(\cos(8y))\right]$$
$$= 64x(-8\sin(8y))$$
$$= -512x\sin(8y)$$

Finally, we evaluate at the point $(5, 10)$:

$$f_{xx}(5,10) = \boxed{0}$$
$$f_{xy}(5,10) = 64\cos(8\cdot 10) \approx \boxed{-7.06}$$
$$f_{yy}(5,10) = -512(5)\sin(8\cdot 10) \approx \boxed{2544.35}$$

NOTE 53. Keep your calculator in radians unless explicitly told to use degrees.

5. Find the second order derivatives of $f(x, y) = ye^{\sin x}$.

Solution: Write

$$f_x(x,y) = \frac{\partial}{\partial x}(ye^{\sin x})$$
$$= y\left[\frac{\partial}{\partial x}(\sin x)\right]\cdot e^{\sin x}$$
$$= y(\cos x)e^{\sin x}$$
$$f_y(x,y) = \frac{\partial}{\partial y}(ye^{\sin x}) = e^{\sin x}$$

Next,

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial}{\partial x}(\underbrace{y(\cos x)e^{\sin x}}_{f_x}) \\ &= y\cos x\left[\frac{\partial}{\partial x}(e^{\sin x})\right] + e^{\sin x}\left[\frac{\partial}{\partial x}(y\cos x)\right] \\ &= y(\cos^2 x)e^{\sin x} - y(\sin x)e^{\sin x} \\ f_{xy}(x,y) &= \frac{\partial}{\partial y}(\underbrace{y(\cos x)e^{\sin x}}_{f_x}) = (\cos x)e^{\sin x} \\ f_{yy}(x,y) &= \frac{\partial}{\partial y}(\underbrace{e^{\sin x}}_{f_y}) = 0 \end{aligned}$$

Thus,

$$\boxed{f_{xx} = y(\cos^2 x)e^{\sin x} - y(\sin x)e^{\sin x}, \quad f_{xy} = (\cos x)e^{\sin x}, \quad f_{yy} = 0}$$

NOTE 54. This homework is not too tough conceptually but it is still difficult because you need to be very careful with your algebra. Keeping track of all the little components is tricky and takes a lot of practice. When you are working on your homework, keep your work neat and detailed. If you are sure you have added each detail at each step, then checking for errors is significantly easier. It is important that you learn how to proofread your own work.

2. Additional Examples

EXAMPLES.

1. Find $f_{xy}(1,2)$ given

$$f(x,y) = \frac{3x\ln(3xy)}{4y}.$$

Round your answer to 4 decimal places.

Solution: We first find f_{xy}, and then evaluate at $(x,y) = (1,2)$. By Clairaut's theorem (Fact 52), the order of differentiation does not matter. We begin by differentiating with respect to x:

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(\frac{3x\ln(3xy)}{4y}\right) \\ &= \frac{1}{4y}\frac{\partial}{\partial x}(3x\ln(3xy)) \\ &= \frac{1}{4y}\left((3x)\frac{\partial}{\partial x}\ln(3xy) + \left(\frac{\partial}{\partial x}(3x)\right)\ln(3xy)\right) \\ &= \frac{1}{4y}\left((3x)\left(\frac{3y}{3xy}\right) + 3\ln(3xy)\right) \end{aligned}$$

$$= \frac{1}{4y}\left(3 + 3\ln(3xy)\right)$$

Next, we differentiate with respect to y:

$$\begin{aligned}
f_{xy} &= \frac{\partial}{\partial y}\left(\frac{1}{4y}\left(3 + 3\ln(3xy)\right)\right) \\
&= \frac{1}{4y}\frac{\partial}{\partial y}\left(3 + 3\ln(3xy)\right) + \frac{\partial}{\partial y}\left(\frac{1}{4y}\right)\left(3 + 3\ln(3xy)\right) \\
&= \frac{1}{4y}\left(\frac{3(3x)}{3xy}\right) - \frac{1}{4y^2}(3 + 3\ln(3xy)) \\
&= \frac{3}{4y^2} - \frac{3 + 3\ln(3xy)}{4y^2} \\
&= -\frac{3\ln(3xy)}{4y^2}
\end{aligned}$$

Next, we evaluate at $(1, 2)$:

$$\begin{aligned}
f_{x,y}(1, 2) &= -\frac{3\ln(3(1)(2))}{4(2)^2} \\
&= -\frac{3\ln(6)}{16} \\
&\approx \boxed{-.3360}
\end{aligned}$$

2. Find $f_{xy}(x, y)$ if

$$f(x, y) = (5x^3 + 3y^2)e^{-xy}.$$

Solution: Write

$$\begin{aligned}
f_x &= \frac{\partial}{\partial x}((5x^3 + 3y^2)e^{-xy}) \\
&= (5x^3 + 3y^2)\frac{\partial}{\partial x}(e^{-xy}) + \left(\frac{\partial}{\partial x}(5x^3 + 3y^2)\right)e^{-xy} \\
&= (5x^3 + 3y^2)(-ye^{-xy}) + (15x^2)e^{-xy} \\
&= (-5x^3y + 3y^3)e^{-xy} + 15x^2e^{-xy} \\
&= (-5x^3y + 15x^2 - 3y^3)e^{-xy}
\end{aligned}$$

Finally,

$$f_{xy} = \frac{\partial}{\partial y}((-5x^3y + 15x^2 - 3y^3)e^{-xy})$$

$$= (-5x^3y + 15x^2 - 3y^3)\frac{\partial}{\partial y}(e^{-xy}) + \left(\frac{\partial}{\partial y}(-5x^3y + 15x^2 - 3y^3)\right) e^{-xy}$$

$$= (-5x^3y + 15x^2 - 3y^3)(-xe^{-xy}) + (-5x^3 - 9y^2)e^{-xy}$$

$$= (5x^4y - 15x^3 + 3xy^3)e^{-xy} + (-5x^3 - 9y^2)e^{-xy}$$

$$= \boxed{(5x^4y - 20x^3 + 3xy^3 - 9y^2)e^{-xy}}$$

3. Compute the second order derivatives of

$$f(x, y) = 10ye^{\cos(5x-3)}.$$

Solution: By second order derivatives, we mean f_{xx}, f_{yy}, f_{xy}. We start by finding f_x and f_y. Write

$$f_x = \frac{\partial}{\partial x}(10ye^{\cos(5x-3)})$$

$$= 10y\frac{\partial}{\partial x}e^{\cos(5x-3)}$$

$$= 10y\left(\frac{\partial}{\partial x}\cos(5x-3)\right)e^{\cos(5x-3)}$$

$$= 10y(5)(-\sin(5x-3))e^{\cos(5x-3)}$$

$$= -50y\sin(5x-3)e^{\cos(5x-3)}$$

$$f_y = \frac{\partial}{\partial y}(10ye^{\cos(5x-3)})$$

$$= 10e^{\cos(5x-3)}\frac{\partial}{\partial y}(y)$$

$$= 10e^{\cos(5x-3)}$$

Next, we take the second partial derivatives

$$f_{xx} = \frac{\partial}{\partial x}\underbrace{(-50y\sin(5x-3)e^{\cos(5x-3)})}_{f_x}$$

$$= -50y\frac{\partial}{\partial x}(\sin(5x-3)e^{\cos(5x-3)})$$

$$= -50y\left[\sin(5x-3)\frac{\partial}{\partial x}e^{\cos(5x-3)} + \left(\frac{\partial}{\partial x}\sin(5x-3)\right)e^{\cos(5x-3)}\right]$$

$$= -50y\left[\sin(5x-3)\left(\frac{\partial}{\partial x}\cos(5x-3)\right)e^{\cos(5x-3)} + 5\cos(5x-3)e^{\cos(5x-3)}\right]$$

$$= -50y\left[-5\sin^2(5x-3)e^{\cos(5x-3)} + 5\cos(5x-3)e^{\cos(5x-3)}\right]$$

$$= 250y\left[\sin^2(5x-3) - \cos(5x-3)\right]e^{\cos(5x-3)}$$

$$f_{xy} = \frac{\partial}{\partial y}\underbrace{\left(-50y\sin(5x-3)e^{\cos(5x-3)}\right)}_{f_x} = -50\sin(5x-3)e^{\cos(5x-3)}$$

$$f_{yy} = \frac{\partial}{\partial y}\underbrace{10e^{\cos(5x-3)}}_{f_y} = 0$$

We conclude

$$\boxed{f_{xx} = 250y\left[\sin^2(5x-3) - \cos(5x-3)\right]e^{\cos(5x-3)}, \quad f_{xy} = -50\sin(5x-3)e^{\cos(5x-3)}, \quad f_{yy} = 0}$$

Lesson 21: Differentials of Multivariable Functions

1. Quick Review of Differentials

Ex 1. Consider the function $f(x) = \sqrt{x}$. We know that $f(9) = \sqrt{9} = 3$, but what is $f(9.1) = \sqrt{9.1}$? Obviously, if you have a calculator this is easy. But instead of just using a calculator, we'll use differentials.

Let $x = 9$ and $x + \Delta x = 9.1$, that is, $\Delta x = .1$. Δx is the *actual change* in the input x. Our goal is to approximate how this change in the input affects the output function, that is, $f(9.1) = f(x + \Delta x)$. For this, we use calculus. Write

$$\Delta y = f(x + \Delta x) - f(x) = f(9.1) - f(9) = \sqrt{9.1} - \sqrt{9}.$$

Δy is the *actual change* the function $f(x)$, which is our goal. In an ideal world, we could compute this directly for any given Δx. But, in general, this is difficult to compute even with a calculator so we settle for an *approximation* of Δy instead.

Observe that the equation

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{12}$$

looks a lot like a derivative. In fact, the only difference between equation (12) and an actual derivative is that we need to take the limit as $\Delta x \to 0$. Because limits deal with things getting really close together, if our Δx is small we can make an approximation of $\dfrac{\Delta y}{\Delta x}$ using equation (12). We can write this like

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \approx f'(x) = \frac{dy}{dx}.$$

More helpfully, we have

$$\Delta y \approx f'(x)\Delta x. \tag{13}$$

This just means that we can approximate the change in the function by taking the change in the input and multiplying it by the derivative of the function. Let's apply this to the example above. Since $f(x) = \sqrt{x}$, we have

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Hence, by equation (13),

$$\sqrt{9.1} - \sqrt{9} = \Delta y \approx f'(9)\Delta x = \frac{1}{2\sqrt{9}}(.1) = \frac{.1}{2(3)} = \frac{1}{60}.$$

So, if $\sqrt{9.1} - \sqrt{9} \approx \frac{1}{60}$, we can add $\sqrt{9}$ to both sides to get

$$\sqrt{9.1} \approx \underbrace{\sqrt{9}}_{3} + \frac{1}{60} \approx 3.01666667$$

Using a calculator, we find

$$\sqrt{9.1} \approx 3.0166207.$$

So our approximation is pretty good.

NOTE 55. We call dx and dy **differentials**. By the nature of derivatives (because we would assume that $\Delta x \to 0$), the smaller Δx is, the better the approximation of Δy.

Think of Δ as the actual change and d as the **infinitesimal** change. This is why we use dx in an integral and not Δx because Δx is "too" big.

2. Differentials of Multivariable Functions

We can apply much of this thinking to functions of more than 1 variable as well. This time, however, we consider how changes in x and y affect $z = f(x, y)$. Our notation will be essentially the same and our goal will be to approximate

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

The **total differential** is given by

$$\partial z = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)dx + f_y(x, y)dy.$$

We can use this formula to approximate Δz (remember, Δz is the **actual** change in z). Now, we take $\Delta x = dx$ and $\Delta y = dy$ and use these to approximate Δz as follows:

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y. \tag{14}$$

We call this equation the **incremental approximation formula for functions of two variables**.

EX 2. Suppose we have $z = f(x, y) = \sqrt{x^2 + y^2}$. Then if $x = 3, y = 4$,

$$f(3, 4) = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

What if we wanted to find $f(3.1, 3.8)$? Take $\Delta x = 3.1 - 3 = .1$ and $\Delta y = 3.8 - 4 = -.2$. Next, note that

$$f_x(x, y) = \frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2}\right) = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}$$
$$f_y(x, y) = \frac{\partial}{\partial y}\left(\sqrt{x^2 + y^2}\right) = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}.$$

Therefore, by equation (14) above,

$$\begin{aligned}\Delta z &\approx \frac{x}{\sqrt{x^2+y^2}}\Delta x + \frac{y}{\sqrt{x^2+y^2}}\Delta y \\ &= \frac{3}{\sqrt{(3)^2+(4)^2}}(.1) + \frac{4}{\sqrt{(3)^2+(4)^2}}(-.2) \\ &= \frac{3}{5}(.1) + \frac{4}{5}(-.2) \\ &= \frac{3}{50} - \frac{8}{50} \\ &= -\frac{5}{50} = -\frac{1}{10} = -.1\end{aligned}$$

So we can write

$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= f(3.1, 3.8) - f(3,4)\end{aligned}$$

and adding $f(3,4)$ to both sides, we get

$$f(3.1, 3.8) = f(3,4) + \Delta z \approx 5 + (-.1) = 4.9.$$

Plugging it into a calculator, $\sqrt{(3.1)^2+(3.8)^2} \approx 4.9041$. So our approximation wasn't too far off.

3. Solution to In-Class Examples

EXAMPLE 1. Use increments to estimate the change in z at $(1,-1)$ if $\frac{\partial z}{\partial x} = 3x+y$ and $\frac{\partial z}{\partial y} = 9y$ given $\Delta x = .01$ and $\Delta y = .02$.

Solution: We use our incremental approximation formula.

$$\begin{aligned}\Delta z &\approx \frac{\partial z}{\partial x}(1,-1)\Delta x + \frac{\partial z}{\partial y}(1,-1)\Delta y \\ &= \underbrace{[3(1)+(-1)]}_{\frac{\partial z}{\partial x}(1,-1)}\underbrace{(.01)}_{\Delta x} + \underbrace{[9(-1)]}_{\frac{\partial z}{\partial y}(1,-1)}\underbrace{(.02)}_{\Delta y} \\ &= 2(.01) - 9(.02) \\ &= .02 - 9(.02) \\ &= -8(.02) = \boxed{-.16}.\end{aligned}$$

EXAMPLE 2. Suppose that when a babysitter feeds a child x donuts and y pieces of cake, the child needs to run $\sqrt{x^2y+7}$ laps in the backyard to be able to go to bed before the parents get home. If one evening the babysitter gives the child 3 donuts and 2 pieces of cake and the next time babysitting, 3.5 donuts and 1.5 pieces of cake, estimate the difference in the number of laps the child will need to run.

Solution: Take $x = 3, y = 2$. Then $\Delta x = 3.5 - 3 = .5$ and $\Delta y = 1.5 - 2 = -.5$. Next, we need to find the derivatives with respect to x and y. Write

$$\frac{\partial z}{\partial x} = \frac{2xy}{2\sqrt{x^2y+7}} = \frac{xy}{\sqrt{x^2y+7}}$$
$$\frac{\partial z}{\partial y} = \frac{x^2}{2\sqrt{x^2y+7}}.$$

Thus,

$$\begin{aligned}
\Delta z &\approx \frac{xy}{\sqrt{x^2y+7}}\Delta x + \frac{x^2}{2\sqrt{x^2y+7}}\Delta y \\
&= \frac{(3)(2)}{\sqrt{(3)^2(2)+7}}(.5) + \frac{(3)^2}{2\sqrt{(3)^2(2)+7}}(-.5) \\
&= \frac{6}{\sqrt{18+7}}(.5) + \frac{9}{2\sqrt{18+7}}(-.5) \\
&= \frac{3}{\sqrt{25}} - \frac{9}{4\sqrt{25}} \\
&= \frac{3}{5} - \frac{9}{20} = \frac{12}{20} - \frac{9}{20} = \boxed{\frac{3}{20}\text{ laps}}.
\end{aligned}$$

Example 3. A company produces boxes with square bases. Suppose they initially create a box that is 10 cm tall and 4 cm wide but they want to increase the box's height by .5 cm. Estimate how they must change the width so that the box stays the same volume.

Solution: Because we are told these boxes have a square base, the formula for volume is $V = hw^2$ where h is the height and w is the width. We are told $h = 10$, $w = 4$, $\Delta h = .5$ and $\Delta V = 0$ (because we want the volume of the box to stay the same). Now, we know that

$$\frac{\partial V}{\partial h} = w^2 \quad \text{and} \quad \frac{\partial V}{\partial w} = 2wh.$$

So, applying our formula we have

$$\begin{aligned}
&\Delta V \approx \frac{\partial V}{\partial h}\Delta h + \frac{\partial V}{\partial w}\Delta w \\
\Rightarrow \quad &\Delta V = (w^2)\Delta h + (2wh)\Delta w \\
\Rightarrow \quad &0 = [(4)^2](.5) + [2(10)(4)]\Delta w \\
&\quad = 8 + 80\Delta w.
\end{aligned}$$

So we need to solve for Δw given

$$0 = 8 + 80\Delta w.$$

We conclude that $\Delta w = -\dfrac{1}{10}$.

This tells us that the width decreases by $\boxed{\frac{1}{10}\text{ cm}}$.

EXAMPLE 4. Suppose the function $S = W^2F + F^2W$ describes the number of fern spores (in millions) released into the air where F is the number of ferns in an area and W is the speed of the wind in miles per hour. Suppose $F = 56$ and $W = 10$ with maximum errors of 2 ferns and 3 miles per hour. Find the approximate relative percentage error in calculating S. Round your answer to the nearest percent.

Solution: Here, we think of the relative errors as our Δ. Let $\Delta F = \pm 2$ and $\Delta W = \pm 3$. We are essentially trying to figure out how changing the inputs (in the sense of correcting the error) changes the number of spores released. We know that

$$S_F = W^2 + 2FW \quad \text{and} \quad S_W = 2WF + F^2.$$

By our formula,

$$\begin{aligned}\Delta S &= (W^2 + 2FW)\Delta F + (2WF + F^2)\Delta W \\ &= [10^2 + 2(56)(10)](\pm 2) + [2(10)(56) + 56^2](\pm 3) \\ &= \pm(100 + 2(560))(2) \pm (2(560) + 56^2)(3) \\ &= \pm 2440 \pm 12,768.\end{aligned}$$

Now, we need to consider the 4 different possibilities that we get from the $\pm$ signs. Write

$$\begin{aligned}2440 + 12,768 &= 15,208 \\ 2440 - 12,768 &= -10,328 \\ -2440 + 12,768 &= 10,328 \\ -2440 - 12,768 &= -15,208.\end{aligned}$$

To find the maximum error, we're looking for is the largest number in *absolute value*. So we say $\Delta S = 15,208$.

Finally, to determine the relative error, we take

$$\frac{\Delta S}{S} = \frac{15,208}{(10)^2(56) + (56)^2(10)} = \frac{15,208}{36,960} \approx .41147.$$

Thus, our answer is $\boxed{41\%}$.

This tells us that our formula is not very good as a model because small changes in the input (i.e., the errors) lead to large changes in the output.

4. Additional Examples

EXAMPLES.

1. The output at a certain plant is

$$Q(x, y) = 0.08x^2 + 0.12xy + 0.03y^2 \text{ units per day},$$

where x is the number of hours of skilled labor used and y is the number of hours of unskilled labor used. Currently, 30 hours of skilled labor and 190 hours of unskilled labor are used each day. Use calculus to estimate to 1 decimal place the change in daily output if an additional 8.5 hours of skilled labor are used each day, while 8 fewer hours of unskilled labor are used each day.

Solution: We are asked to calculate the change in the daily output, which is ΔQ. We are told that $x = 30$, $y = 190$, $\Delta x = 8.5$, and $\Delta y = -8$. Now,

$$\frac{\partial Q}{\partial x} = 0.16x + 0.12y \quad \text{and} \quad \frac{\partial Q}{\partial y} = 0.12x + 0.06y.$$

So we write

$$\begin{aligned}\Delta Q &\approx \frac{\partial Q}{\partial x}(30, 190)\Delta x + \frac{\partial Q}{\partial y}(30, 190)\Delta y \\ &= (0.16(30) + 0.12(190))(8.5) + (0.12(30) + 0.06(190))(-8) \\ &\approx \boxed{114.6 \text{ units per day}}\end{aligned}$$

2. The productivity of a company is

$$P(x, y) = 30x^{4/5}y^{1/5} \text{ thousands of units}$$

where x is the number of employees and y is the amount of capital expenditure in thousands of dollars. What is the change in the productivity if the number of employees is decreased from 225 to 200 and the capital spent is increased from \$28,000 to \$36,000? Round to 2 decimal places.

Solution: We compute ΔP given that $x = 225$, $y = 28$, $\Delta x = 225 - 200 = 25$, and $\Delta y = 28 - 36 = -8$ (note that we are measuring y in thousands). The change in our variables is always measured from what is the case now to what will be in the future.

Now, differentiating P, we get

$$\begin{aligned}\frac{\partial P}{\partial x} &= 30(4/5)x^{-1/5}y^{1/5} \\ &= 24x^{-1/5}y^{1/5} \\ \frac{\partial P}{\partial y} &= 30(1/5)x^{4/5}y^{-4/5} \\ &= 6x^{4/5}y^{-4/5}\end{aligned}$$

Hence,

$$\begin{aligned}\Delta P &\approx \frac{\partial P}{\partial x}(225, 28)\Delta x + \frac{\partial P}{\partial y}(225, 28)\Delta y \\ &= 24(225)^{-1/5}(28)^{1/5}(25) + 6(225)^{4/5}(36)^{-4/5}(-8) \\ &\approx \boxed{141.25 \text{ thousands of units}}\end{aligned}$$

3. A soft drink can is h centimeters tall and has a radius of r centimeters. The cost of material in the can is 0.001 cents per cm^2 and the cost of the soda

itself is 0.0015 cents per cm^3. The cans are currently 7 cm tall and have a radius of 5 cm. Use calculus to estimate the effect on costs of increasing the radius by 0.5 cm and decreasing the height by 0.9 cm. Round your answer to 3 decimals.

Solution: We assume that the can is a true cylinder so that we can compute its volume and surface area.

The volume of a cylinder is given by $V = \pi r^2 h$ and the surface area of a cylinder is given by $SA = 2\pi r^2 + 2\pi rh$. Thus, the cost of the material is described by

$$\begin{aligned} C(r,h) &= 0.001(2\pi r^2 + 2\pi rh) + 0.0015(\pi r^2 h) \\ &= 0.002\pi r^2 + 0.002\pi rh + 0.0015\pi r^2 h \end{aligned}$$

We want to find ΔC given that $r = 5$, $h = 7$, $\Delta r = 0.5$, and $\Delta h = -0.9$. Differentiating,

$$\begin{aligned} \frac{\partial C}{\partial r} &= 0.004\pi r + 0.002\pi h + 0.003\pi rh \\ \frac{\partial C}{\partial h} &= 0.002\pi r + 0.0015\pi r^2 \end{aligned}$$

Thus,

$$\begin{aligned} \Delta C &\approx \frac{\partial C}{\partial r}(5,7)\Delta r + \frac{\partial C}{\partial h}(5,7)\Delta h \\ &= [0.004\pi(5) + 0.002\pi(7) + 0.003\pi(5)(7)](0.5) + [0.002\pi(5) + 0.0015\pi(5)^2](-0.9) \\ &= [0.02\pi + 0.014\pi + 0.105\pi](0.5) + [0.01\pi + 0.0375\pi](-0.9) \\ &= [0.139\pi](0.5) + [0.0475\pi](-0.9) \\ &= 0.0695\pi - .04275\pi \\ &\approx \boxed{0.084 \text{ cents per can}} \end{aligned}$$

Lesson 22: Chain Rule, Functions of Several Variables

1. Chain Rule for Multivariable Functions

Sometimes functions are written as function of more than one variable but can actually come down to a single variable, which is to say that x and y are related through a *third* variable.

This situation often occurs when both of our variables are related to time. For example, to know how the weather feels to humans, we need the actual temperature and the humidity. But both of these variables depend on the time of day we are considering — even if the temperature and humidity are otherwise unrelated.

Ex 1. Suppose $x(t) = t^4 + 1$ describes the number of socks a store sells over time and $y(t) = 3t^2 + 6$ describes the price of the socks over time. Let $z(t) = xy$ be the revenue the store earns from the sale of the socks. How does the revenue change with respect to time?

One way to do this is to write z entirely in terms of t and then differentiate, but the more complicated x, y, and z become, the more difficult this task is. Instead, we use the **chain rule for multivariable functions**:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial z}{\partial y}\left(\frac{dy}{dt}\right) \tag{15}$$

Note 56. Observe that this is $\frac{dz}{dt}$ and **not** $\frac{\partial z}{\partial t}$. As a function *of* t, z is a function of a *single variable*. The proper notation, then, is $\frac{dz}{dt}$.

In this Ex, we have

$$\frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial y} = x, \quad \frac{dx}{dt} = 4t^3, \quad \frac{dy}{dt} = 6t.$$

Applying equation (15), we see that the change in the revenue over time is given by

$$\frac{dz}{dt} = (y)(4t^3) + (x)(6t).$$

We leave it like this (this is how you should input your answer for the homework).

Examples.

1. Find $\frac{dz}{dt}$ given

$$z = x^2y^2, \quad x = \cos t, \quad y = 3t^3.$$

Solution: Differentiating,

$$\frac{\partial z}{\partial x} = 2xy^2, \quad \frac{\partial z}{\partial y} = 2x^2y, \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = 9t^2.$$

Applying equation (15),

$$\frac{dz}{dt} = \underbrace{(2xy^2)}_{\frac{\partial z}{\partial x}}\underbrace{(-\sin t)}_{\frac{dx}{dt}} + \underbrace{(2x^2y)}_{\frac{\partial z}{\partial y}}\underbrace{(9t^2)}_{\frac{dy}{dt}} = \boxed{-2xy^2\sin(t) + 18x^2yt^2}.$$

Again, leave it like this for the homework.

2. Given $z = \sqrt{x^2+y^2}$, $x = \ln\sqrt{t}$, and $y = \frac{1}{t}$, find $\frac{dz}{dt}$ evaluated at $t = 1$.

Solution: We start this problem in the same manner as above: we differentiate. Write

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}, \quad \frac{dx}{dt} = \frac{1}{2t}, \quad \frac{dy}{dt} = -\frac{1}{t^2}.$$

And as before, we apply equation (15),

$$\frac{dz}{dt} = \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\frac{\partial z}{\partial x}}\underbrace{\left(\frac{1}{2t}\right)}_{\frac{dx}{dt}} + \underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\frac{\partial z}{\partial y}}\underbrace{\left(-\frac{1}{t^2}\right)}_{\frac{dy}{dt}}.$$

Now, we are asked to evaluate at $t = 1$, which means wherever we see t, we put 1 instead. But what do we do about x and y? We return to our original expressions for x and y,

$$x = \ln\sqrt{t} \quad \text{and} \quad y = \frac{1}{t},$$

and take $t = 1$. So,

$$x(1) = \ln\sqrt{1} = \ln(1) = 0 \quad \text{and} \quad y(1) = \frac{1}{1} = 1.$$

Therefore,

$$\begin{aligned}\frac{dz}{dt}(t=1) &= \frac{x}{\sqrt{x^2+y^2}}\left(\frac{1}{2t}\right) + \frac{y}{\sqrt{x^2+y^2}}\left(-\frac{1}{t^2}\right)\\ &= \frac{0}{\sqrt{(0)^2+(1)^2}}\left(\frac{1}{2(1)}\right) + \frac{1}{\sqrt{(0)^2+(1)^2}}\left(-\frac{1}{(1)^2}\right)\\ &= \boxed{-1}.\end{aligned}$$

3. The width of a box with a square base is increasing at a rate of 2 in/min while height is decreasing at a rate of 10 in/min. What is the rate of change of the surface area when the width is 18 inches and the height is 30 inches?

Solution: Let w be the width and h be the height of the box. Because the box has a square base, the surface area is given by

$$SA = 2w^2 + 4wh.$$

Our goal is to find $\frac{dSA}{dt}$ when $w = 18$ and $h = 30$. So we need to use equation (15) and then evaluate at $(w, h) = (18, 30)$. Now, we are told that

$$\frac{dw}{dt} = 2 \quad \text{and} \quad \frac{dh}{dt} = -10$$

since the width is *increasing* at a rate of 2 in/min and the height is *decreasing* at a rate of 10 in/min. Further,

$$\frac{\partial SA}{\partial w} = 4w + 4h \quad \text{and} \quad \frac{\partial SA}{\partial h} = 4w.$$

By equation (15), we have

$$\frac{dSA}{dt} = \frac{\partial SA}{\partial w}\left(\frac{dw}{dt}\right) + \frac{\partial SA}{\partial h}\left(\frac{dh}{dt}\right) = [4w + 4h](2) + [4w](-10).$$

Evaluating at $(w, h) = (18, 30)$, we get

$$\frac{dSA}{dt}(18, 30) = [4(18) + 4(30)](2) + [4(18)](-10) = \boxed{-336 \text{ in/min}}.$$

4. $PV = nRT$ is the ideal gas law where P is pressure in Pascals (Pa), V is volume in liters (L), and T is temperature in Kelvin (K) of n moles of gas. R is the ideal gas constant. Suppose P is decreasing at a rate of 1 Pa/min and the temperature is increasing at a rate of 2 K/min. How is the volume changing?

Solution: We are tasked with finding $\frac{dV}{dt}$. We were given the formula

$$PV = nRT \Rightarrow V = \frac{nRT}{P}.$$

Now, n and R are constants and P, T are variables. Written in this form, V is a function of P and T. So,

$$\frac{\partial V}{\partial P} = \frac{\partial}{\partial P}\left(\frac{nRT}{P}\right) = nRT\left[\frac{\partial}{\partial P}\left(\frac{1}{P}\right)\right] = nRT\left(-\frac{1}{P^2}\right) = -\frac{nRT}{P^2}$$

and

$$\frac{\partial V}{\partial T} = \frac{\partial}{\partial T}\left(\frac{nRT}{P}\right) = \frac{nR}{P}\left[\frac{\partial}{\partial T}(T)\right] = \frac{nR}{P}.$$

Since we are told that P is *decreasing* at a rate of 1 Pa/min and the temperature is *increasing* at a rate of 2 K/min, we have that

$$\frac{dP}{dt} = -1 \quad \text{and} \quad \frac{dT}{dt} = 2.$$

By equation (15),

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial P}\frac{dP}{dt} + \frac{\partial V}{\partial T}\frac{dT}{dt} \\ &= \left(-\frac{nRT}{P^2}\right)(-1) + \left(\frac{nR}{P}\right)(2) \\ &= \boxed{\frac{nRT}{P^2} + \frac{2nR}{P}}. \end{aligned}$$

We leave our answer in the above form because they have not given us enough information to be more specific.

NOTE 57. This lesson may seem very similar to Lesson 21 and rightly so as the only difference is that the incremental approximation formula is an approximation of the chain rule for multivariable functions. When do we know which formula applies?

Think of incremental change as change over some period of time and the type of change discussed in this lesson as instantaneous (that is, a derivative). If the questions asks about a "rate of change" or something implying instantaneous change, then the chain rule for multivariable functions applies. However, if the question asks for an estimate or approximation or talks about change not in terms of a derivative function, then the incremental approximation formula applies.

2. Additional Example

EXAMPLES.

1. The monthly demand for the Instant Pie Maker is given by

$$D(x, y) = \frac{9}{1000} x e^{xy/1000} \text{ units},$$

where x dollars are spend on infomercials and y dollars are spent on in-person demonstrations. If t months from now, $x = 95 + t^{2/3}$ dollars are spent on infomercials and $y = t\ln(1+t)$ dollars are spent on demonstrations, at approximately what rate will the demand be changing with respect to time 8 months from now? Round your answer to 3 decimal places.

Solution: We need to use the product rule to find our partial derivative with respect to x:

$$\begin{aligned} \frac{\partial D}{\partial x} &= \frac{9}{1000} x \left(\frac{\partial}{\partial x} e^{xy/1000}\right) + \left(\frac{\partial}{\partial x}\frac{9}{1000}x\right) e^{xy/1000} \\ &= \frac{9}{1000} x \left(\frac{y}{1000}\right) e^{xy/1000} + \frac{9}{1000} e^{xy/1000} \end{aligned}$$

$$= \frac{9xy}{(1000)^2}e^{xy/1000} + \frac{9}{1000}e^{xy/1000}$$
$$= \frac{9xy + 9000}{(1000)^2}e^{xy/1000}$$

Next, we find the partial derivative with respect to y:

$$\frac{\partial D}{\partial y} = \frac{\partial}{\partial y}\left(\frac{9}{1000}xe^{xy/1000}\right)$$
$$= \left(\frac{9}{1000}x\right)\left(\frac{x}{1000}\right)e^{xy/1000}$$
$$= \frac{9x^2}{(1000)^2}e^{xy/1000}$$

Moreover,

$$\frac{dx}{dt} = \frac{2}{3}t^{-1/3}$$
$$\frac{dy}{dt} = t\left(\frac{1}{1+t}\right) + \ln(1+t) = \frac{t}{1+t} + \ln(1+t)$$

We are evaluating at $t = 8$, which means we have

$$x(8) = 95 + (8)^{2/3} = 99$$
$$\frac{dx}{dt}(8) = \frac{2}{3}(8)^{-1/3} = \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3}$$
$$y(8) = 8\ln(9)$$
$$\frac{dy}{dt}(8) = \frac{8}{9} + \ln(9)$$

Putting this together, we have

$$\frac{dz}{dt} = \frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt}$$
$$= \frac{9(99)(8\ln(9)) + 9000}{(1000)^2}e^{99(8\ln(9))/1000}\left(\frac{1}{3}\right) + \frac{9(99)^2}{(1000)^2}e^{99(8\ln(9))/1000}\left(\frac{8}{9} + \ln(9)\right)$$
$$\approx \boxed{1.589}$$

2. Use the chain rule to compute $\dfrac{dz}{dt}$ at $t = .6$ given

$$z = x\sin(3y), \quad x = e^{1.25t}, \quad y = \pi - 9t.$$

Round to 4 decimal places.

Solution: Write

$$\frac{\partial z}{\partial x} = \sin(3y) \qquad \frac{\partial z}{\partial y} = 3x\cos(3y)$$

$$\frac{dx}{dt} = 1.25e^{1.25t} \qquad \frac{dy}{dt} = -9$$

Since we are asked to evaluate at $t = .6$,

$$x(.6) = e^{1.25(.6)}$$
$$y(.6) = \pi - 9(.6)$$

Putting this together,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= \sin(3(\pi - 9(.6)))\left(1.25e^{1.25(.6)}\right) + 3(e^{1.25(.6)})\cos(3(\pi - 9(.6)))(-9) \\ &= \sin(3\pi - 16.2)(1.25e^(.75)) - 27\cos(3\pi - 16.2) \\ &\approx \boxed{-51.6285}\end{aligned}$$

Lesson 23: Extrema of Functions of Two Variables (I)

1. Extrema of Multivariable Functions

Just like with functions of a single variable, we consider how to find the minima and maxima (plural of minimum and maximum, respectively) of functions of several variables. We call the minima and maxima the **extrema** (plural of extremum).

DEFINITION 58.

- A **local (relative) minimum point** is a point (x, y) such that the function is the smallest in some region about (x, y)
- A **local (relative) maximum point** is a point (x, y) such that the function is the largest in some region about (x, y)

There is also a notion of a **global (absolute) minimum/maximum point**, which is the point (x, y) that makes the function the smallest/largest on the whole graph. We do not address this concept in this class.

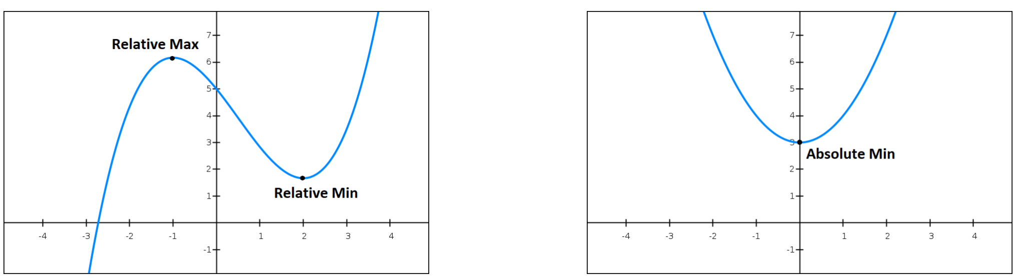


FIGURE 8. Functions of a Single Variable

DEFINITION 59.

- A **local (relative) minimum** is the smallest function value in some area.
- A **local (relative) maximum** is the largest function value in some area.

Observe

$$\begin{array}{ccc} \text{extrema points} & & \text{extrema} \\ (x, y) & \longleftrightarrow & f(x, y) \\ \text{ordered pair} & & \text{function value} \end{array},$$

and so extrema points are ordered pairs while extrema are function values.

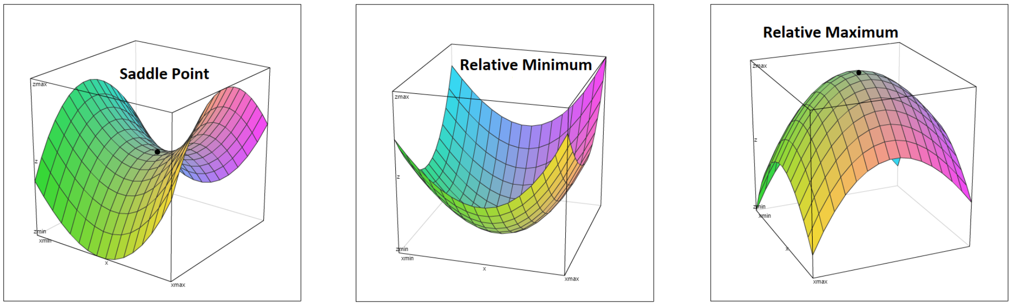


FIGURE 9. Functions of a Several Variables

EX 1. A difference between functions of a single variable and functions of several variables is that functions of several variables can have what are called “saddle points”.

DEFINITION 60. The **critical points** of a function $f(x, y)$ are the ordered pairs (x_0, y_0) such that

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0).$$

DEFINITION 61. The function

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$$

is called the **discriminant** of $f(x, y)$.

Second Derivative Test: Suppose (x_0, y_0) is a critical point of f. If

(1) $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, (x_0, y_0) is a local maximum point

(2) $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, (x_0, y_0) is a local minimum point

(3) $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point

(4) $D(x_0, y_0) = 0$, the test is inconclusive (i.e., this test doesn't give you any information)

EXAMPLES.

1. Find and classify the critical points of

$$f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} - y - x.$$

Solution: We apply the following steps.

Step 1: Find critical points

Critical points are points (x_0, y_0) that make **both** f_x and f_y equal to 0. Write

$$f_x(x, y) = x^2 - 1 = (x - 1)(x + 1)$$

$$f_y(x, y) = y^2 - 1 = (y - 1)(y + 1)$$

Hence, if $f_x = 0$, then

$$(x - 1)(x + 1) = 0 \quad \Rightarrow \quad x = \pm 1$$

and if $f_y = 0$, then

$$(y-1)(y+1) = 0 \quad \Rightarrow \quad y = \pm 1.$$

So, setting both f_x and f_y equal to zero, our critical points are

$$(x_0, y_0) = (1,1),\ (1,-1),\ (-1,1),\ (-1,-1).$$

Step 2: Find second derivatives

Write

$$f_{xx} = 2x, \quad f_{yy} = 2y, \text{ and } f_{xy} = 0.$$

Step 3: Find discriminant

Our formula for the discriminant is

$$D = f_{xx} f_{yy} - (f_{xy})^2.$$

So,

$$D(x,y) = \underset{\substack{\uparrow \\ f_{xx}}}{(2x)}\ \underset{\substack{\uparrow \\ f_{yy}}}{(2y)} - (\underset{\substack{\uparrow \\ f_{xy}}}{0})^2 = 4xy.$$

Step 4: Apply test

We go through each critical point and apply the second derivative test.

Critical Point	$D(x_0, y_0)$	$f_{xx}(x_0, y_0)$	Classification
$(1,1)$	$4(1)(1) = 4 > 0$	$2(1) = 2 > 0$	local min
$(1,-1)$	$4(1)(-1) = -4 < 0$	$---$	saddle point
$(-1,1)$	$4(-1)(1) = -4 < 0$	$---$	saddle point
$(-1,-1)$	$4(-1)(-1) = 4 > 0$	$2(-1) < 0$	local max

2. Find and classify the critical points of

$$g(u,v) = u^2 v - uv - v^2.$$

Solution: Again, we go through our steps.

Step 1: Find critical points

We have

$$g_u = 2uv - v = v(2u-1)$$
$$g_v = u^2 - u - 2v$$

Recall that our critical points are the (u_0, v_0) that make **both** g_u and g_v equal to 0. We write

$$0 = g_u = v(2u-1).$$

Here, we have a choice: either $v = 0$ **or** $u = \frac{1}{2}$. Moreover, we have

$$0 = g_v = u^2 - u - 2v \quad \Rightarrow \quad 2v = u^2 - u.$$

At this point, it is not clear what our points (u_0, v_0) should be. This is where we break into cases:

Case 1. $v = 0$

If $v = 0$, then $2v = u^2 - u$ becomes

$$0 = u^2 - u = u(u-1).$$

So $u = 0, 1$. This means that two of our critical points are

$$(\underset{\stackrel{\uparrow}{u}}{0}, \underset{\stackrel{\uparrow}{v}}{0}) \quad \text{and} \quad (\underset{\stackrel{\uparrow}{u}}{1}, \underset{\stackrel{\uparrow}{v}}{0}).$$

(Because we were given $g(u, v)$, the order is going to be (u, v).)

Case 2. $u = \frac{1}{2}$

If $u = \frac{1}{2}$, then $2v = u^2 - u$ becomes

$$2v = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4},$$

which implies

$$v = -\frac{1}{8}.$$

Hence, our last critical point is

$$\left(\frac{1}{2}, -\frac{1}{8}\right).$$

Putting this all together, our critical points are

$$(0,0), \quad (1,0), \quad \text{and} \quad \left(\frac{1}{2}, -\frac{1}{8}\right).$$

Step 2: Find second derivatives

We have

$$g_{uu} = 2v, \quad g_{vv} = -2, \quad \text{and} \quad g_{uv} = 2u - 1.$$

Step 3: Find discriminant

The formula for the discriminant is given by

$$D = g_{uu}g_{vv} - (g_{uv})^2$$

which becomes

$$D(u,v) = \underset{\stackrel{\uparrow}{g_{uu}}}{(2v)}\,\underset{\stackrel{\uparrow}{g_{vv}}}{(-2)} - (\underbrace{2u-1}_{g_{uv}})^2 = -4v - (2u-1)^2.$$

Step 4: Apply test

Write

$$D\left(\frac{1}{2},-\frac{1}{8}\right) = -4\left(-\frac{1}{8}\right) - \left(2\left(\frac{1}{2}\right)-1\right)^2 = \frac{1}{2}$$

$$g_{uu}\left(\frac{1}{2},-\frac{1}{8}\right) = 2\left(-\frac{1}{8}\right) = -\frac{1}{4}$$

$$D(0,0) = -4(0) - (2(0)-1)^2 = -1$$

$$D(1,0) = -4(0) - (2(1)-1)^2 = -1$$

Critical Point	$D(u_0, v_0)$	$g_{xx}(u_0, v_0)$	Classification
$\left(\frac{1}{2},-\frac{1}{8}\right)$	$\frac{1}{2} > 0$	$-\frac{1}{4} < 0$	local max
$(0,0)$	$-1 < 0$	$---$	saddle point
$(1,0)$	$-1 < 0$	$---$	saddle point

3. Find the local minima and maxima of

$$f(x,y) = x^2 + y^2 - 2x + 2y.$$

Solution: Observe that this question is different than the previous 2 examples. Before we were asked to classify the *critical points*, but now we are asked to find the actual *function values* at the critical points. Our process, fortunately, doesn't change too much. We still need to find and classify the critical points *but then* we need to plug them back into $f(x,y)$ to determine the function value.

Step 1: Find critical points

We have

$$f_x = 2x - 2 \quad \text{and} \quad f_y = 2y + 2.$$

Thus,

$$0 = f_x = 2x - 2 \quad \Rightarrow \quad x = 1$$

and

$$0 = f_y = 2y + 2 \quad \Rightarrow \quad y = -1.$$

This means our critical point is $(1,-1)$.

Step 2: Find second derivatives

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

Step 3: Find discriminant

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (2)(2) - (0)^2 = 4$$

Step 4: Apply test

Since $D(1,-1) = 4 > 0$ and $f_{xx}(1,-1) = 2 > 0$, we have a **local min** at $(1,-1)$ and **no local max**.

Step 5: Determine function values

By the previous steps, we know that

$$f(1,-1) = (1)^2 + (-1)^2 - 2(1) + 2(-1) = 1 + 1 - 2 - 2 = \boxed{-2}$$

is a local minimum of $f(x,y)$ (recall that a min or max is a *function value*).

4. Count the number of minima, maxima, and saddle points of

$$g(x,y) = -\frac{x^3}{3} + 2xy - \frac{y^2}{2}.$$

Solution: We go through our steps and then plug our points to find the function values.

Step 1: Find critical points

$$g_x = -x^2 + 2y \quad \text{and} \quad g_y = 2x - y$$

So

$$0 = g_x = -x^2 + 2y \quad \Rightarrow \quad x^2 = 2y$$

and

$$0 = g_y = 2x - y \quad \Rightarrow \quad 2x = y.$$

Since we have that $y = 2x$, we can substitute this into $2y = x^2$ which means

$$2(2x) = x^2 \Rightarrow 4x = x^2 \quad \Rightarrow \quad x = 0 \textbf{ or } x = 4.$$

We break this down into cases.

Case 1. $x = 0$

If $x = 0$, then $y = 2(0) = 0$. Hence, one critical point is $(0,0)$.

Case 2. $x = 4$

If $x = 4$, then $y = 2(4) = 8$. Thus, another critical point is $(4,8)$.

Putting this together, our critical points are

$$(0,0) \quad \text{and} \quad (4,8).$$

Step 2: Find second derivatives

$$g_{xx} = -2x, \quad g_{yy} = -1, \quad \text{and} \quad g_{xy} = 2$$

Step 3: Find discriminant

$$D(x,y) = g_{xx}g_{yy} - (g_{xy})^2 = (-2x)(-1) - (2)^2 = 2x - 4$$

Step 4: Apply test

We write

$$D(0,0) = 2(0) - 4 = -4$$

$$D(4,8) = 2(4) - 4 = 8 - 4 = 4$$

$$g_{xx}(4,8) = -2(4) = -8$$

Hence,

Critical Point	$D(x_0, y_0)$	$g_{xx}(x_0, y_0)$	Classification
$(0,0)$	$-4 < 0$	$---$	saddle point
$(4,8)$	$4 > 0$	$-8 < 0$	local max

Hence, we have 0 minimum, 1 maximum, and 1 saddle point.

2. Additional Examples

EXAMPLES.

1. Find all the local minima and maxima points of

$$f(x,y) = 3x^2 - xy + 7y^2 - 8x - 54y - 5.$$

Solution: We go through our steps.

Step 1: Find critical points

We have

$$f_x = 6x - y - 8$$
$$f_y = -x + 14y - 54$$

Setting f_x equal to 0,

$$0 = 6x - y - 8$$
$$\Rightarrow \quad y = 6x - 8$$

Setting f_y equal to 0,

$$0 = -x + 14y - 54$$
$$\Rightarrow \quad 54 = -x + 14\underbrace{(6x - 8)}_{y}$$
$$= -x + 84x - 112$$
$$\Rightarrow \quad 166 = 83x$$
$$\Rightarrow \quad x = 2$$
$$\Rightarrow \quad y = 6(2) - 8 = 4$$

Thus, our critical point is $(2, 4)$.

Step 2: Find second derivatives

Write

$$f_{xx} = 6, \quad f_{yy} = 14, \quad \text{and} \quad f_{xy} = -1.$$

Step 3: Find discriminant

The discriminant is given by

$$D = f_{xx}f_{yy} - (f_{xy})^2.$$

So,

$$D(x,y) = \underset{f_{xx}}{\underset{\uparrow}{(6)}}\,\underset{f_{yy}}{\underset{\uparrow}{(14)}} - (\underset{f_{xy}}{\underset{\uparrow}{-1}})^2 = 84 - 1 = 83.$$

Step 4: Apply test

We see that $D(2,4) > 0$ and $f_{xx}(2,4) > 0$ which means that f has a $\boxed{\text{minima point at } (2,4)}$.

2. Find and classify the critical points of $g(x,y) = x^2 + xy + \frac{1}{32}y^4 - 8$ where

$$g_x = 2x + y \quad \text{and} \quad g_y = x + \frac{1}{8}y^3.$$

Solution: We go through our steps.

Step 1: Find critical points

Setting g_x and g_y equal to 0, we have

$$
\begin{aligned}
0 &= \underbrace{2x + y}_{g_x} \\
\Rightarrow \quad y &= -2x \\
0 &= \underbrace{x + \frac{1}{8}y^3}_{g_y} \\
&= x + \frac{1}{8}(\underset{y}{\underset{\uparrow}{-2x}})^3 \\
&= x + \frac{1}{8}(-8x^3) \\
&= x - x^3 \\
&= x(1 - x^2)
\end{aligned}
$$

Now, we have three possible solutions to $0 = x(1 - x^2)$, either

$$x = 0, \quad x = 1, \quad \text{or } x = -1.$$

We check each of these cases.

Case 1. $x = 0$

If $x = 0$, the $y = -2(0) = 0$. We conclude one critical point is $(0,0)$.

Case 2. $x = 1$

If $x = 1$, then $y = -2(1) = -2$. Hence, another critical point is $(1,-2)$.

Case 3. $x = -1$

If $x = -1$, then $y = -2(-1) = 2$. Thus, our final critical point is $(-1, 2)$.

Step 2: Find second derivatives

Write

$$g_{xx} = 2, \quad g_{yy} = \frac{3}{8}y^2, \quad g_{xy} = 1.$$

Step 3: Find discriminant

Our formula for the discriminant is

$$D = f_{xx}f_{yy} - (f_{xy})^2.$$

So,

$$D(x,y) = \underset{g_{xx}}{\underset{\uparrow}{(2)}} \underset{g_{yy}}{\underset{\uparrow}{\left(\frac{3}{8}y^2\right)}} - (\underset{g_{xy}}{\underset{\uparrow}{1}})^2 = \frac{3}{4}y^2 - 1.$$

Step 4: Apply test

Critical Point	$D(x_0, y_0)$	$f_{xx}(x_0, y_0)$	Classification
$(0,0)$	$-1 < 0$	$---$	saddle point
$(1,-2)$	$2 > 0$	$2 > 0$	local min
$(-1,2)$	$2 > 0$	$2 > 0$	local min

3. Find and classify the critical points of

$$f(x,y) = 16x^4 + 8x + 12y^3 - y + 7.$$

Solution:

Step 1: Find critical points

We have

$$f_x = 64x^3 + 8$$

$$f_y = 36y^2 - 1$$

Setting f_x and f_y equal to 0, we have

$$\begin{aligned} 0 &= 64x^3 + 8 \\ \Rightarrow \quad -8 &= 64x^3 \\ \Rightarrow \quad -\frac{1}{8} &= x^3 \\ \Rightarrow \quad \sqrt[3]{-\frac{1}{8}} &= x \\ \Rightarrow \quad x &= -\frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} 0 &= 36y^2 - 1 \\ \Rightarrow \quad 1 &= 36y^2 \\ \Rightarrow \quad \frac{1}{36} &= y^2 \\ \Rightarrow \quad \pm\frac{1}{6} &= y \end{aligned}$$

Hence, we see our critical points are

$$\left(-\frac{1}{2}, \frac{1}{6}\right) \quad \text{and} \quad \left(-\frac{1}{2}, -\frac{1}{6}\right).$$

Step 2: Find second derivatives

Write

$$f_{xx} = 192x^2, \quad f_{yy} = 72y, \quad , f_{xy} = 0.$$

Step 3: Find discriminant

The formula for the discriminant is

$$D = f_{xx}f_{yy} - (f_{xy})^2.$$

So,

$$D(x, y) = \underset{\substack{\uparrow \\ f_{xx}}}{(192x^2)}\underset{\substack{\uparrow \\ f_{yy}}}{(72y)} - (\underset{\substack{\uparrow \\ f_{xy}}}{0})^2 = 13{,}824x^2y.$$

Step 4: Apply test

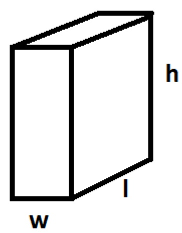
Critical Point	$D(x_0, y_0)$	$g_{xx}(x_0, y_0)$	Classification
$\left(-\frac{1}{2}, \frac{1}{6}\right)$	$576 > 0$	$48 > 0$	local min
$\left(-\frac{1}{2}, -\frac{1}{6}\right)$	$-576 < 0$	$---$	saddle point

Lesson 24: Extrema of Functions of Two Variables (II)

1. Solutions to In-Class Examples

EXAMPLE 1. We are tasked with constructing a rectangular box with a volume of 64 cubic feet. The material for the top costs 8 dollars per square foot, the material for the sides costs 10 dollars per square foot, and the material for the bottom costs 4 dollars per square foot. To the nearest cent, what is the minimum cost for such a box? (Round your answer to 2 decimal places.)

Solution: Let w be the width, h be the height, and l be the length of this box.



The volume of the box is given by $V = whl$. The goal here is to minimize the cost function, **not** the volume. In fact, we are requiring that the volume be exactly 64 cubic feet. We'll call this the constraint.

By the picture above, you should see the area of the top is lw, the area of the bottom is lw, and the total area of the sides is $2wh + 2lh$ (this is because we are **not** assuming the box has a square base so w and l may be different). Thus, by the information we are given above, our cost function is

$$C(w,h,l) = 8\underbrace{(lw)}_{\substack{\text{area of}\\\text{top}}} + 10\underbrace{(2wh+2lh)}_{\substack{\text{area of}\\\text{sides}}} + 4\underbrace{(lw)}_{\substack{\text{area of}\\\text{bottom}}} = 12lw + 20wh + 20lh.$$

Unfortunately, this is a function of 3 variables and our tools only work for functions of 2 variables. To resolve this issue, we use the constraint ($lwh = 64$) to rewrite the cost function as a function of 2 variables.

Write $l = \dfrac{64}{wh}$. Substituting, we get

$$\begin{aligned} C(w,h) &= 12\left(\frac{64}{wh}\right)w + 20wh + 20\left(\frac{64}{wh}\right)h \\ &= \frac{768}{h} + 20wh + \frac{1280}{w}. \end{aligned}$$

Since this is now a function of 2 variables, we can find the critical points.

Differentiating,

$$C_w = 20h - \frac{1280}{w^2} \quad \text{and} \quad C_h = -\frac{768}{h^2} + 20w.$$

Recall that the critical points are the points (w, h) that make both C_w and C_h equal to zero. So

$$\begin{aligned} & C_w = 0 \\ \Rightarrow \quad & 20h - \frac{1280}{w^2} = 0 \\ \Rightarrow \quad & 20h = \frac{1280}{w^2} \\ \Rightarrow \quad & w^2 h = \frac{1280}{20} = 64 \end{aligned}$$

and

$$\begin{aligned} & C_h = 0 \\ \Rightarrow \quad & 20w - \frac{768}{h^2} = 0 \\ \Rightarrow \quad & 20w = \frac{768}{h^2} \\ \Rightarrow \quad & h^2 w = \frac{768}{20} = \frac{192}{5}. \end{aligned}$$

Now, we observe that $\frac{192}{5} = \frac{3}{5}(64)$. This means that

$$h^2 w = \frac{192}{5} = \frac{3}{5}(64) = \frac{3}{5} w^2 h.$$

Since we are assuming the volume is 64 cubic inches, we must have $w \neq 0$ and $h \neq 0$. So, we divide both sides by hw and our equation becomes

$$h = \frac{3}{5} w.$$

Then, because $w^2 h = 64$,

$$w^2 \left(\frac{3}{5} w \right) = 64 \quad \Rightarrow \quad w^3 = \frac{5(64)}{3} = \frac{320}{3}.$$

Thus,

$$w = \sqrt[3]{\frac{320}{3}}$$

and so

$$h = \frac{3}{5} w = \frac{3}{5} \sqrt[3]{\frac{320}{3}}.$$

Our critical point is then

$$(w,h) = \left(\sqrt[3]{\frac{320}{3}}, \frac{3}{5}\sqrt[3]{\frac{320}{3}}\right).$$

Therefore, our cost is minimized at

$$C\left(\sqrt[3]{\frac{320}{3}}, \frac{3}{5}\sqrt[3]{\frac{320}{3}}\right) = \frac{768}{\frac{3}{5}\sqrt[3]{\frac{320}{3}}} + 20\left(\sqrt[3]{\frac{320}{3}}\right)\left(\frac{3}{5}\sqrt[3]{\frac{320}{3}}\right) + \frac{1280}{\sqrt[3]{\frac{320}{3}}}$$
$$\approx \boxed{\$809.695}$$

Note 62. Technically, you should check that this is actually a minimum by going through the Second Derivative Test. But since this is the only critical point and, in this context, it is much easier to increase a cost rather than minimize it, we can assume that this is indeed a minimum. For word problems, if there is a single critical point, then this is probably the point we need to find — else the problem itself is poorly posed.

Example 2. The post office will accept packages whose combined length and girth is at most 50 inches (girth is the total perimeter around the package perpendicular to the length and the length is the largest of the 3 dimensions). What is the largest volume that can be sent in a rectangular box? (Round your answer to the nearest integer.)

Solution: Our goal is to maximize the volume $V = lwh$ where l is the largest of the 3 dimensions. The girth is then $g = 2w + 2h$. We are told that the combined length and girth can be at most 50 inches, But, to maximize the volume, we will want to maximize the girth as well (think about why this is true). Set $l + g = 50$.

Now, our volume function is a function of more than 2 variables so we need to rewrite it in terms of 2 variables. We know that $l + g = 50$ and that $g = 2w + 2h$, so

$$l = 50 - g = 50 - \underbrace{(2w + 2h)}_{g} = 50 - 2w - 2h.$$

Thus, our volume function becomes

$$V = lwh = (\underbrace{50 - 2w - 2h}_{l})wh = 50wh - 2wh^2 - 2w^2h.$$

This is the function we want to maximize. We find its critical points.

Find critical points: $V_w = 50h - 2h^2 - 4wh$ which means

$$0 = V_w = 50h - 2h^2 - 4wh = h(50 - 2h - 4w).$$

So either $h = 0$ **or** $50 - 2h - 4w = 0$. Because we can't let one of our dimensions be 0, we throw out the case where $h = 0$. From $50 - 2h - 4w = 0$, we get

$$\begin{aligned} & 0 = 50 - 2h - 4w \\ \Rightarrow\ & 0 = 25 - h - 2w \\ \Rightarrow\ & h = 25 - 2w \end{aligned}$$

$V_h = 50w - 4wh - 2w^2$ means

$$0 = V_h = 50w - 4wh - 2w^2 = w(50 - 4h - 2w),$$

so either $w = 0$ **or** $50 - 4h - 2w = 0$. Again, we can't let one of our dimensions be 0 so we throw out $w = 0$. Hence, we have $50 - 4h - 2w = 0$ which means

$$\begin{aligned} 0 &= 50 - 4h - 2w \\ \Rightarrow \ 0 &= 25 - 2h - w \\ \Rightarrow \ w &= 25 - 2h \\ &= 25 - 2\underbrace{(25 - 2w)}_{h} \\ &= 25 - 50 + 4w \\ &= -25 + 4w \\ \Rightarrow \ -3w &= -25 \\ \Rightarrow \ w &= \frac{25}{3}. \end{aligned}$$

Thus,

$$h = 25 - 2w = 25 - 2\underbrace{\left(\frac{25}{3}\right)}_{w} = 25 - \frac{2(25)}{3} = \frac{25}{3}.$$

Our critical point is then $(w, h) = \left(\frac{25}{3}, \frac{25}{3}\right)$.

Finally, we need to plug this into our function for volume. Since $l = 50 - 2w - 2h$,

$$l = 50 - 2\left(\frac{25}{3}\right) - 2\left(\frac{25}{3}\right) = \frac{150}{3} - \frac{50}{3} - \frac{50}{3} = \frac{50}{3}.$$

Thus, our maximum volume is

$$V = \underbrace{\left(\frac{50}{3}\right)}_{l}\underbrace{\left(\frac{25}{3}\right)}_{w}\underbrace{\left(\frac{25}{3}\right)}_{h} = \frac{31,250}{27} \approx \boxed{1157 \text{ in}^3}.$$

Example 3. A biologist must make a medium to grow a type of bacteria. The percentage of salt in the medium is given by $S = 0.01x^2y^2z$, where x, y, and z are amounts in liters of 3 different nutrients mixed together to create the medium. The ideal salt percentage for this type of bacteria is 48%. The costs of x, y, and z nutrient solutions are respectively, 6, 3, and 8 dollars per liter. Determine the minimum cost that can be achieved. (Round your answer to the nearest 2 decimal places.)

Solution: We want to minimize the cost function, given by

$$C(x, y, z) = 6x + 3y + 8z$$

such that $S(x, y, z) = .48$. We want to reduce our cost function to a function of 2 variables, which we do using S. Write

$$.48 = 0.01x^2y^2z \quad \Rightarrow \quad z = \frac{48}{x^2y^2}.$$

Substituting this into our cost function, we get

$$C(x, y) = 6x + 3y + 8\underbrace{\left(\frac{48}{x^2y^2}\right)}_{z} = 6x + 3y + \frac{384}{x^2y^2}.$$

Our next step is to find the critical points.

Find critical points: $C_x = 6 - \dfrac{2(384)}{x^3y^2} = 6 - \dfrac{768}{x^3y^2}$, so $C_x = 0$ implies

$$\begin{aligned} 0 &= 6 - \frac{768}{x^3y^2} \\ \Rightarrow \quad \frac{768}{x^3y^2} &= 6 \\ \Rightarrow \quad 128 &= x^3y^2 \end{aligned}$$

$C_y = 3 - \dfrac{768}{x^2y^3}$, so $C_y = 0$ implies

$$\begin{aligned} 0 &= 3 - \frac{768}{x^2y^3} \\ \Rightarrow \quad \frac{768}{x^2y^3} &= 3 \\ \Rightarrow \quad 256 &= x^2y^3 \end{aligned}$$

Now, since $2(128) = 256$, we see by our work above that

$$2\underbrace{(x^3y^2)}_{128} = \underbrace{x^2y^3}_{256},$$

Hence, either $x = 0$ **or** $y = 0$ **or** $2x = y$. Because we want $S(x, y, z) = .48 \neq 0$, we can't have $x = 0$ or $y = 0$. So we have $2x = y$.

Since

$$\begin{aligned} 256 &= x^2y^3 \\ \Rightarrow \quad 256 &= x^2(\underset{\substack{\uparrow \\ y}}{2x})^3 = 8x^5 \\ \Rightarrow \quad 32 &= x^5 \\ \Rightarrow \quad x &= 2 \end{aligned}$$

Then $y = 2(2) = 4$ implies our critical point is then $(2, 4)$.

The function value at $(2,4)$ is

$$C(2,4) = 6(2) + 3(4) + \frac{384}{(2)^2(4)^2} = 12 + 12 + \frac{384}{64} = 12 + 12 + 6 = \boxed{30}.$$

EXAMPLE 4. A manufacturer is planning to sell a new product at the price of 310 dollars per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, consumers will buy approximately $\dfrac{270y}{y+4} + \dfrac{300x}{x+9}$ units of the product. If manufacturing costs for the product are 220 dollars per unit, how much should the manufacturer spend on development and how much on promotion to generate the largest possible profit? Round your answer to the nearest cent.

Solution: Profit is the difference of revenue and cost. Here, the revenue is $.31\left(\dfrac{270y}{y+4} + \dfrac{300x}{x+9}\right)$ because it is the number of units sold times their price and we are measuring our dollars in thousands. The cost is $.22\left(\dfrac{270y}{y+4} + \dfrac{300x}{x+9}\right) + x + y$ because the cost is the cost of each unit times the number sold but we also need to consider what is spent on development (x) and promotion (y). Thus, the function we want to maximize is

$$\begin{aligned}
\text{Profit} = P(x,y) &= .31\left(\frac{270y}{y+4} + \frac{300x}{x+9}\right) - \left[.22\left(\frac{270y}{y+4} + \frac{300x}{x+9}\right) + x + y\right] \\
&= .31\left(\frac{270y}{y+4} + \frac{300x}{x+9}\right) - .22\left(\frac{270y}{y+4} + \frac{300x}{x+9}\right) - x - y \\
&= .09\left(\frac{270y}{y+4} + \frac{300x}{x+9}\right) - x - y \\
&= \frac{24.3y}{y+4} + \frac{27x}{x+9} - x - y
\end{aligned}$$

To maximize $P(x,y)$, we need to find its critical points.

Find the critical points: Differentiating with respect to x, we get

$$\begin{aligned}
P_x &= \frac{(x+9)(27) - (27x)(1)}{(x+9)^2} - 1 \\
&= \frac{27x + 243 - 27x}{(x+9)^2} - 1 \\
&= \frac{243}{(x+9)^2} - 1
\end{aligned}$$

and, with respect to y,

$$P_y = \frac{(y+4)(24.3) - (24.3y)(1)}{(y+4)^2} - 1$$

$$= \frac{24.3y + 97.2 - 24.3y}{(y+4)^2} - 1$$
$$= \frac{97.2}{(y+4)^2} - 1$$

Setting P_x and P_y equal to 0, we see that

$$0 = P_x = \frac{243}{(x+9)^2} - 1$$
$$\Rightarrow \quad 1 = \frac{243}{(x+9)^2}$$
$$\Rightarrow \quad (x+9)^2 = 243$$
$$\Rightarrow x = \pm\sqrt{243} - 9$$

and

$$0 = P_y = \frac{97.2}{(y+4)^2} - 1$$
$$\Rightarrow \quad 1 = \frac{97.2}{(y+4)^2}$$
$$\Rightarrow \quad (y+4)^2 = 97.2$$
$$\Rightarrow \quad y = \pm\sqrt{97.2} - 4.$$

Since it doesn't make sense for x or y to be negative, we conclude our critical point is

$$(x, y) = (\sqrt{243} - 9, \sqrt{97.2} - 4).$$

This tells us that the developer should spend 6588.46 dollars on development and 5859.00 dollars on promotion since x and y are measured in thousands. Be sure to multiply by 1000 and *then* round.

2. Additional Examples

EXAMPLES.

1. Find the number of minima, maxima, and saddle points of

$$f(x, y) = -20 + 2y + 6x^2y + 6x^2 + \frac{7}{4}y^2.$$

Solution: We go through our steps.

Step 1: Find critical points

We have

$$f_x = 12xy + 12x$$
$$f_y = 2 + 6x^2 + \frac{7}{2}y$$

Setting f_x equal to 0, we have

$$0 = 12xy + 12x = 12x(y + 1).$$

Hence, we see that either $x = 0$ or $y = -1$. We break this into cases.

Case 1. $x = 0$

Since $0 = 2 + 6x^2 + \frac{7}{2}y = 2 + 6(0)^2 + \frac{7}{2}y$, we have

$$\begin{aligned} 0 &= 2 + \frac{7}{2}y \\ \Rightarrow \quad -2 &= \frac{7}{2}y \\ \Rightarrow \quad -\frac{4}{7} &= y \end{aligned}$$

So, one critical point is $\left(0, -\frac{4}{7}\right)$.

Case 2. $y = -1$

Since $0 = 2 + 6x^2 + \frac{7}{2}y = 2 + 6x^2 + \frac{7}{2}(-1)$, we have

$$\begin{aligned} 0 &= 2 + 6x^2 - \frac{7}{2} \\ &= -\frac{3}{2} + 6x^2 \\ \Rightarrow \quad \frac{3}{2} &= 6x^2 \\ \Rightarrow \quad \frac{3}{12} &= x^2 \\ \Rightarrow \quad \frac{1}{4} &= x^2 \\ \Rightarrow \quad \pm\frac{1}{2} &= x \end{aligned}$$

Thus, our other critical points are $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, -1\right)$.

Step 2: Find second derivatives

Write

$$f_{xx} = 12y + 12, \quad f_{yy} = \frac{7}{2}, \quad f_{xy} = 12x.$$

Step 3: Find discriminant

The formula for the discriminant is given by

$$D = f_{xx}f_{yy} - (f_{xy})^2.$$

So,

$$D(x,y) = \underbrace{(12y+12)}_{f_{xx}}\underset{\substack{\uparrow \\ f_{yy}}}{\left(\frac{7}{2}\right)} - (\underset{\substack{\uparrow \\ f_{xy}}}{12x})^2 = 42y + 42 - 144x^2$$

Step 4: Apply test

Critical Point	$D(x_0, y_0)$	$f_{xx}(x_0, y_0)$	Classification
$\left(0, -\frac{4}{7}\right)$	$18 > 0$	$\frac{36}{7} > 0$	local min
$\left(\frac{1}{2}, -1\right)$	$-36 < 0$	$---$	saddle point
$\left(-\frac{1}{2}, -1\right)$	$-36 < 0$	$---$	saddle point

2. In a certain experiment to find out the ideal studying conditions in a library, the performance of the subject is influenced by two types of stimulus, noise and temperature, measured in positive units of x and y, respectively. Their performance is measured by the function

$$f(x,y) = 16 + 7xye^{6-7x^2-8y^2}.$$

How many units of each stimulus results in the maximum performance? Round your answer to 4 decimal places.

<u>**Solution**</u>: We need to find the critical points of f, keeping in mind that we want $x, y > 0$. The the derivative with respect to either variable will require both the chain rule and the product rule. Starting by differentiating with respect to x, write

$$\begin{aligned}
f_x &= \frac{\partial}{\partial x}(16 + 7xye^{6-7x^2-8y^2}) \\
&= 7xy\left[\frac{\partial}{\partial x}e^{6-7x^2-8y^2}\right] + \left[\frac{\partial}{\partial x}7xy\right]e^{6-7x^2-8y^2} \\
&= 7xy\left[\frac{\partial}{\partial x}(6-7x^2-8y^2)\right]e^{6-7x^2-8y^2} + 7ye^{6-7x^2-8y^2} \\
&= 7xy(-14x)e^{6-7x^2-8y^2} + 7ye^{6-7x^2-8y^2}
\end{aligned}$$

$$= (-98x^2 + 7)ye^{6-7x^2-8y^2}$$

Next,

$$\begin{aligned}
f_y &= \frac{\partial}{\partial y}(16 + 7xye^{6-7x^2-8y^2}) \\
&= 7xy\left[\frac{\partial}{\partial y}e^{6-7x^2-8y^2}\right] + \left[\frac{\partial}{\partial y}7xy\right]e^{6-7x^2-8y^2} \\
&= 7xy\left[\frac{\partial}{\partial y}(6-7x^2-8y^2)\right]e^{6-7x^2-8y^2} + 7xe^{6-7x^2-8y^2} \\
&= 7xy(-16y)e^{6-7x^2-8y^2} + 7xe^{6-7x^2-8y^2} \\
&= (-112y^2 + 7)xe^{6-7x^2-8y^2}
\end{aligned}$$

Observe that $e^{6-7x^2-8y^2} \neq 0$ for any value of x, y. Hence, if we set f_x, f_y equal to zero, we may immediately simplify:

$$\begin{aligned}
f_x = (-98x^2 + 7)ye^{6-7x^2-8y^2} &= 0 \\
\Rightarrow \quad (-98x^2 + 7)y &= 0 \\
f_y = (-112y^2 + 7)xe^{6-7x^2-8y^2} &= 0 \\
\Rightarrow \quad (-112y^2 + 7)x &= 0
\end{aligned}$$

Now, if $(-98x^2 + 7)y = 0$, then either

$$-98x^2 + 7 = 0 \quad \text{or} \quad y = 0.$$

If $-98x^2 + 7 = 0$, then

$$\begin{aligned}
0 &= -98x^2 + 7 \\
\Rightarrow \quad 98x^2 &= 7 \\
\Rightarrow \quad x^2 &= \frac{1}{14} \\
\Rightarrow \quad x &= \pm\sqrt{\frac{1}{14}}
\end{aligned}$$

Because we want $x, y > 0$, we conclude that

$$x = \sqrt{\frac{1}{14}}.$$

Next, if $(-112y^2 + 7)x = 0$, then either

$$-112y^2 + 7 = 0 \quad \text{or} \quad x = 0.$$

If $-112y^2 + 7 = 0$, then

$$\begin{aligned} 0 &= -112y^2 + 7 \\ \Rightarrow \quad 112y^2 &= 7 \\ \Rightarrow \quad y^2 &= \frac{1}{16} \\ \Rightarrow \quad y &= \pm\sqrt{\frac{1}{16}} \end{aligned}$$

Since we assume that $x, y > 0$, we have

$$y = \sqrt{\frac{1}{16}}.$$

Therefore, our critical point is

$$\left(\sqrt{\frac{1}{14}}, \sqrt{\frac{1}{16}}\right) = \boxed{(.2673, .2500)}.$$

Lesson 25: Lagrange Multipliers - Constrained Min/Max (I)

1. Lagrange Multipliers

Lagrange multipliers is another method of finding minima and maxima of functions of more than one variable. This method applies when we are finding extrema that is subject to some **constraint**.

> The Method of Lagrange Multipliers: Suppose we want to minimize or maximize a function $f(x, y)$ subject to the constraint $g(x, y) = C$. Introduce a "dummy" variable, λ, and solve the system of equations
>
> $$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y) \\ f_y(x, y) &= \lambda g_y(x, y) \\ g(x, y) &= C \end{aligned}$$
>
> for (x, y).

Remark 63. In this setup, our method only works for functions of 2 variables. If a problem is presented with more than 2 variables or if we are asked to classify critical points, then we need to use the method from the previous lessons. However, if the problem contains key words like "subject to" or has only 2 variables and a constraint, then this method applies.

Ex 1. Maximize the area of a rectangular garden subject to the constraint that its perimeter is 100 ft.

Solution: Let x be the length and y the width of the garden. Then the function we are maximizing is

$$f(x, y) = xy.$$

But this is subject to the constraint that

$$\underbrace{2x + 2y}_{\text{perimeter}} = 100.$$

By our method, we set up our system of equations:

$$\begin{aligned} \underset{f_x}{\underset{\uparrow}{y}} &= \lambda\underset{g_x}{\underset{\uparrow}{(2)}} = 2\lambda \\ \underset{f_y}{\underset{\uparrow}{x}} &= \lambda\underset{g_y}{\underset{\uparrow}{(2)}} = 2\lambda \\ \underbrace{2x + 2y}_{g(x,y)} &= \underbrace{100}_{C} \end{aligned}$$

We solve for x and y (which means we eliminate λ).

Since $x = 2\lambda$ and $y = 2\lambda$, we see that given $2x + 2y = 100$,

$$100 = 2(\underset{\uparrow\atop x}{2\lambda}) + 2(\underset{\uparrow\atop y}{2\lambda}) = 4\lambda + 4\lambda = 8\lambda.$$

Thus, $\lambda = \dfrac{100}{8} = \dfrac{25}{2}$ which means

$$x = 2\underbrace{\left(\frac{25}{2}\right)}_{\lambda} = 25 \quad \text{and} \quad y = 2\underbrace{\left(\frac{25}{2}\right)}_{\lambda} = 25.$$

We conclude that the area is maximized when $x = 25$ and $y = 25$ and the maximum area of the garden is $25(25) = 625$ ft^2.

Question: How do we know this is a maximum and not a minimum? If $x = 1$, $y = 49$, then the area is 49 which is certainly less than 625. Hence, in this context, we can safely conclude that this is a maximum. For these problems, you *always* need to consider whether your answer makes sense in context.

NOTE 64. Lagrange multipliers will never tell you if there is a saddle point because that involves classifying critical points. Critical points are different than solutions to the system of equations for Lagrange multipliers.

EXAMPLES.

1. Minimize $f(x, y) = (x + 1)^2 + (y - 2)^2$ subject to $g(x, y) = x^2 + y^2 = 125$.

Solution: Taking derivatives, we see that

$$f_x = 2(x + 1), \quad f_y = 2(y - 2), \quad g_x = 2x, \quad g_y = 2y.$$

Setting up our equations

$$\boxed{\begin{aligned} 2(x + 1) &= \lambda(2x) = 2\lambda x \\ 2(y - 2) &= \lambda(2y) = 2\lambda y \\ x^2 + y^2 &= 125 \end{aligned}}$$

The method of Lagrange multipliers calls for a little creativity and the key is staying flexible. Focus on the first two equations. We have

$$2(x + 1) = 2\lambda x \quad \Rightarrow \quad x + 1 = \lambda x$$
$$2(y - 2) = 2\lambda y \quad \Rightarrow \quad y - 2 = \lambda y$$

Thus,

$$\begin{aligned} & x + 1 = \lambda x & & y - 2 = \lambda y \\ \Rightarrow\ & x - \lambda x + 1 = 0 & \Rightarrow\ & y - \lambda y - 2 = 0 \\ \Rightarrow\ & x(1 - \lambda) + 1 = 0 & \Rightarrow\ & y(1 - \lambda) - 2 = 0 \\ \Rightarrow\ & x(1 - \lambda) = -1 & \Rightarrow\ & y(1 - \lambda) = 2 \end{aligned}$$

From this we gather that $x, y \neq 0$ else these equations can't be true. This means that we may divide by x and y without losing solutions to the system. **Always** be careful when dividing through by a variable. If you don't know that the variable is non-zero, then you will lose solutions.

Next,

$$-\frac{1}{x} = 1 - \lambda = \frac{2}{y},$$

and so, in particular, $-\frac{1}{x} = \frac{2}{y}$. By cross-multiplication this becomes

$$-y = 2x \quad \Rightarrow \quad y = -2x.$$

By our constraint, $g(x, y) = x^2 + y^2 = 12$. We evaluate at $y = -2x$:

$$\begin{aligned} 125 &= x^2 + y^2 \\ &= x^2 + (-2x)^2 \\ &= x^2 + 4x^2 \\ &= 5x^2 \\ \Rightarrow \quad 25 &= x^2 \end{aligned}$$

We conclude $x = \pm 5$, which implies $y = -2(\pm 5) = \mp 10$. Thus, our extrema points are

$$(5, -10) \quad \text{and} \quad (-5, 10).$$

Evaluating $f(x, y)$ at these points,

$$f(5, -10) = (5+1)^2 + (-10-2)^2 = 6^2 + (-12)^2 = 36 + 144 = 180 \longleftarrow \text{Max}$$

$$f(-5, 10) = (-5+1)^2 + (10-2)^2 = (-4)^2 + (8)^2 = 16 + 64 = 80 \longleftarrow \text{Min}$$

Therefore, the minimum is $\boxed{80}$ because it is the smaller of the two values.

2. Find the minimum value of $x^2 e^{y^2}$ subject to $2y^2 + 2x = 6$.

Solution: Whatever function we are minimizing or maximizing is our $f(x, y)$ and the constraint is $g(x, y)$. Thus, we have

$$f(x, y) = x^2 e^{y^2} \quad \text{and} \quad g(x, y) = 2y^2 + 2x = 6.$$

Next, we find derivatives:

$$f_x = 2xe^{y^2}, \quad f_y = 2x^2 y e^{y^2}, \quad g_x = 2, \quad g_y = 4y.$$

We set up our equations to get

$$\boxed{\begin{aligned} 2xe^{y^2} &= 2\lambda \\ 2x^2 y e^{y^2} &= 4\lambda y \\ 2y^2 + 2x &= 6 \end{aligned}}$$

By the first equation, we see that

$$2xe^{y^2} = 2\lambda \quad \Rightarrow \quad xe^{y^2} = \lambda.$$

Substituting this into the second equation,

$$2yx^2e^{y^2} = 4\underbrace{\left(xe^{y^2}\right)}_{\lambda}y.$$

We may divide both sides by $2e^{y^2}$ because this is **never** 0. Thus, our equation becomes

$$x^2y = 2xy.$$

Subtracting $2xy$ from both sides

$$x^2y - 2xy = 0 \quad \Rightarrow \quad (x^2 - 2x)y = 0.$$

Hence, either $y = 0$ **or**

$$x^2 - 2x = 0 \quad \Rightarrow \quad x(x-2) = 0 \quad \Rightarrow \quad x = 0 \textbf{ or } x = 2.$$

We check all three cases:

Case 1. $y = 0$

If $y = 0$, then our constraint implies that

$$0 + 2x = 6 \quad \Rightarrow \quad x = 3.$$

Thus, one solution is $(3, 0)$.

Case 2. $x = 0$

If $x = 0$, then by our constraint:

$$2y^2 + 0 = 6 \quad \Rightarrow \quad y^2 = 3 \quad \Rightarrow \quad y = \pm\sqrt{3}.$$

So two of our solutions are $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.

Case 3. $x = 2$

If $x = 2$, then

$$\begin{aligned} 2y^2 + 2(2) &= 6 \\ \Rightarrow \quad 2y^2 &= 2 \\ \Rightarrow \quad y^2 &= 1 \\ \Rightarrow \quad y &= \pm 1. \end{aligned}$$

This adds another two solutions: $(2, 1)$ and $(2, -1)$.

Putting this all together, our solutions are

$$(3, 0), \quad (0, \sqrt{3}), \quad (0, -\sqrt{3}), \quad (2, 1), \quad (2, -1).$$

Finally, we check the function values:

$$f(3, 0) = (3)^2e^{(0)^2} = 9(1) = 9$$

$$f(0, \sqrt{3}) = (0)^2e^{(\sqrt{3})^2} = 0 \longleftarrow \text{min}$$

$$f(0,-\sqrt{3}) = (0)^2 e^{(-\sqrt{3})^2} = 0 \longleftarrow \text{min}$$
$$f(2,1) = (2)^2 e^{(1)^2} = 4e \longleftarrow \text{max}$$
$$f(2,-1) = (2)^2 e^{(-1)^2} = 4e \longleftarrow \text{max}$$

Therefore, the function's minimum value is $\boxed{0}$.

3. Find the minimum value of $f(x,y) = y^2 - x^2 - 4x$ subject to $y = 8 - 2x$.

Solution: Our $f(x,y) = y^2 - x^2 - 4x$ but we need to determine our $g(x,y)$. We are told our constraint is $y = 8 - 2x$ and, adding $2x$ to both sides, we have $2x+y = 8$. Hence, $g(x,y) = 2x+y = 8$. Next, we differentiate:

$$f_x = -2x - 4, \quad f_y = 2y, \quad g_x = 2, \quad g_y = 1.$$

Setting up our equations,

$$\boxed{\begin{aligned} -2x - 4 &= 2\lambda \\ 2y &= \lambda \\ 2x + y &= 8 \end{aligned}}$$

We know immediately that $2y = \lambda$, so, substituting into the first equation, we get

$$-2x - 4 = 2\underbrace{(2y)}_{\lambda} = 4y.$$

Dividing both sides by 4, we get

$$y = -\frac{1}{2}x - 1.$$

According to our constraint,

$$8 = 2x + y = 2x + \underbrace{\left(-\frac{1}{2}x - 1\right)}_{y} = 2x - \frac{1}{2}x - 1 = \frac{3}{2}x - 1.$$

We solve $8 = \frac{3}{2}x - 1$ for x:

$$\begin{aligned} 8 &= \frac{3}{2}x - 1 \\ \Rightarrow \quad 9 &= \frac{3}{2}x \\ \Rightarrow \quad 18 &= 3x \\ \Rightarrow \quad 6 &= x \end{aligned}$$

Since $x = 6$ and $y = -\frac{1}{2}x - 1$, we have $y = -\frac{1}{2}(6) - 1 = -3 - 1 = -4$. Thus, our solution is $(6, -4)$. Plugging this into the function,

$$f(6, -4) = (-4)^2 - (6)^2 - 4(6) = 16 - 36 - 24 = \boxed{-44}.$$

NOTE 65. We should check that this is actually a minimum as opposed to a maximum. To do this, we check **any other point** that satisfies $2x + y = 8$, say $(0, 8)$. If -44 is a minimum, then we must have $-44 < f(0, 8) = 8^2 - (0)^2 - 4(0) = 64$. So we can rest easy knowing that this really is the minimum of the function subject to the given constraint.

4. Find the maximum value of $f(x, y) = \frac{2}{3}x^{3/2}y$ subject to $x = 10 - y$.

Solution: Again, we need to rearrange our constraint to determine our $g(x, y)$. Adding y to both sides of $x = 10 - y$, we get $x + y = 10$ which means

$$g(x, y) = x + y = 10.$$

Next, we differentiate:

$$f_x = x^{1/2}y, \quad f_y = \frac{2}{3}x^{3/2}, \quad g_x = 1, \quad g_y = 1.$$

Now, we set up our equations:

$$\boxed{\begin{aligned} x^{1/2}y &= \lambda \\ \frac{2}{3}x^{3/2} &= \lambda \\ x + y &= 10 \end{aligned}}$$

Since $\lambda = x^{1/2}y$ and $\lambda = \frac{2}{3}x^{3/2}$, we can write

$$x^{1/2}y = \frac{2}{3}x^{3/2}.$$

Subtracting $\frac{2}{3}x^{3/2}$ from both sides, we get

$$\begin{aligned} 0 &= x^{1/2}y - \frac{2}{3}x^{3/2} \\ &= x^{1/2}\left(y - \frac{2}{3}x\right) \end{aligned}$$

This implies we have two solutions, either $x^{1/2} = 0 \Rightarrow x = 0$ **or** $y = \frac{2}{3}x$.

We check both cases.

Case 1. $x = 0$

If $x = 0$, our constraint implies that $0 + y = 10 \Rightarrow y = 10$. Hence, one solution is $(0, 10)$.

Case 2. $\frac{2}{3}x = y$

If $\frac{2}{3}x = y$, then

$$10 = x + y = x + \frac{2}{3}x = \frac{5}{3}x \quad \Rightarrow \quad 10 = \frac{5}{3}x \quad \Rightarrow \quad x = 6.$$

Thus, since $y = \frac{2}{3}(6) = 4$, one solution is $(6, 4)$.

Putting this together, we have two solutions:

$$(0, 10) \quad \text{and} \quad (6, 4).$$

We check the function values at these points:

$$f(0, 10) = \frac{2}{3}(0)^{3/2}(10) = 0 \longleftarrow \text{min}$$
$$f(6, 4) = \frac{2}{3}(6)^{3/2}(4) = 16\sqrt{6} \longleftarrow \text{max}$$

Therefore, the maximum value is $\boxed{16\sqrt{6}}$.

2. Additional Examples

EXAMPLES.

1. Find the extrema of $f(x, y) = e^{-xy}$ subject to $9x^2 + 4y^2 \leq 72$.

Solution: This is a slightly different problem than what we have encountered so far. Here, our constraint is an *inequality* rather than an *equality*. Fortunately, this is not as daunting as it may appear.

We break this problem into two parts: (1) we find the critical points of $f(x, y)$ which are contained in the region described by $g(x, y) = 9x^2 + 4y^2 < 72$ and (2) we apply the Lagrange multiplier method to $f(x, y)$ subject to $g(x, y) = 9x^2 + 4y^2 = 72$.

(1) The derivatives of $f(x, y)$ are

$$f_x = -ye^{-xy} \quad \text{and} \quad f_y = -xe^{-xy}.$$

Setting these equal to 0, we see that $x = 0, y = 0$ because e^{-xy} is **never** 0. Since the point $(0, 0)$ satisfies $g(x, y) = 9x^2 + 4y^2 < 72$, we include this point in our list of solutions.

(2) Now, we assume that $g(x, y) = 9x^2 + 4y^2 = 72$ and apply the Lagrange multiplier method. The derivatives of $g(x, y)$ are

$$g_x = 18x \quad \text{and} \quad g_y = 8y.$$

Setting up our system of equations,

$$\boxed{\begin{aligned} -ye^{-xy} &= 18\lambda x \\ -xe^{-xy} &= 8\lambda y \\ 9x^2 + 4y^2 &= 72 \end{aligned}}$$

Next, we solve for (x, y).

We observe that if any of x, y, or λ are 0, then $x = 0$ and $y = 0$. But this case has already been covered, so we assume that $x, y, \lambda \neq 0$. This means we can divide by x and y to get λ by itself. Write

$$-\frac{y}{18x}e^{-xy} = \lambda \quad \text{and} \quad -\frac{x}{8y}e^{-xy} = \lambda.$$

Thus,

$$-\frac{y}{18x}e^{-xy} = -\frac{x}{8y}e^{-xy}.$$

Because $-e^{-xy}$ is never zero, we may divide through on both sides to get

$$\frac{y}{18x} = \frac{x}{8y}.$$

Cross-multiplying:

$$8y^2 = 18x^2 \quad \Rightarrow \quad y^2 = \frac{18}{8}x^2 = \frac{9}{4}x^2.$$

Now, we return to our constraint and substitute for y^2,

$$\begin{aligned} 72 &= 9x^2 + 4y^2 \\ &= 9x^2 + 4\underbrace{\left(\frac{9}{4}x^2\right)}_{y^2} \\ &= 9x^2 + 9x^2 \\ &= 18x^2. \end{aligned}$$

Solving for x, we find $x = \pm 2$. Since $y^2 = \dfrac{9}{4}x^2$, we get $y^2 = 9 \Rightarrow y = \pm 3$. Therefore, we get the following 4 solutions:

$$(2, 3), \quad (2, -3), \quad (-2, 3), \quad (-2, -3).$$

We need to check the function values at each of our solutions from both (1) and (2):

$$\begin{aligned} f(0, 0) &= e^{-(0)(0)} = 1 \\ f(2, 3) &= e^{-(2)(3)} = e^{-6} \longleftarrow \text{min} \\ f(2, -3) &= e^{-(2)(-3)} = e^{6} \longleftarrow \text{max} \end{aligned}$$

$$f(-2,3) = e^{-(-2)(3)} = e^{6} \longleftarrow \text{max}$$
$$f(-2,-3) = e^{-(-2)(-3)} = e^{-6} \longleftarrow \text{min}$$

Thus, the minimum function value is $\boxed{e^{-6}}$ and the maximum function value is $\boxed{e^{6}}$.

2. Let $f(x,y) = \ln(3xy^2)$. Find the maximum value of the function subject to $5x^2 + 4y^2 = 8$. Round your answer to 4 decimal places.

Solution: We maximize $f(x,y) = \ln(3xy^2)$ subject to the constraint $g(x,y) = 5x^2 + 4y^2 = 8$. Our derivatives are

$$f_x = \frac{3y^2}{3xy^2} = \frac{1}{x}, \quad f_y = \frac{6xy}{3xy^2} = \frac{2}{y}, \quad g_x = 10x, \quad g_y = 8y.$$

Observe that $x > 0$ else f is not defined.

We set up our system of equations:

$$\boxed{\begin{aligned} \frac{1}{x} &= 10\lambda x \\ \frac{2}{y} &= 8\lambda y \\ 5x^2 + 4y^2 &= 8 \end{aligned}}$$

We know that $x, y \neq 0$ else $f(x,y)$ is not defined and that $\lambda \neq 0$ else the first two equations won't be satisfied.

Solving for λ in the first equation, we have

$$\frac{1}{10x^2} = \lambda.$$

Substituting into the second equation,

$$\frac{2}{y} = 8\underbrace{\left(\frac{1}{10x^2}\right)}_{\lambda} y = \frac{4y}{5x^2} \quad \Rightarrow \quad \frac{2}{y} = \frac{4y}{5x^2}.$$

Cross-multiplying, we get

$$10x^2 = 4y^2 \quad \Rightarrow \quad 5x^2 = 2y^2 \quad \Rightarrow \quad x^2 = \frac{2}{5}y^2.$$

Substituting this into the third equation, we have

$$\begin{aligned} 8 &= 5x^2 + 4y^2 \\ &= 5\left(\frac{2}{5}y^2\right) + 4y^2 \\ &= 2y^2 + 4y^2 = 6y^2 \end{aligned}$$

$$\Rightarrow \quad \frac{8}{6} = y^2$$
$$\Rightarrow \quad \frac{4}{3} = y^2$$
$$\Rightarrow \quad \pm\sqrt{\frac{4}{3}} = y$$

Then, since $x^2 = \frac{2}{5}y^2$, we have

$$\begin{aligned} x^2 &= \frac{2}{5}\left(\pm\sqrt{\frac{4}{3}}\right)^2 \\ &= \frac{2}{5}\left(\frac{4}{3}\right) \\ &= \frac{8}{15} \\ \Rightarrow \quad x &= \pm\sqrt{\frac{8}{15}} \end{aligned}$$

But since $x > 0$, we only have two solutions:

$$\left(\sqrt{\frac{8}{15}}, \sqrt{\frac{4}{3}}\right) \quad \text{and} \quad \left(\sqrt{\frac{8}{15}}, -\sqrt{\frac{4}{3}}\right).$$

Finally, we check our function value at these points:

$$\begin{aligned} \ln\left(3\sqrt{\frac{8}{15}}\left(\pm\sqrt{\frac{4}{3}}\right)^2\right) &= \ln\left(3\sqrt{\frac{8}{15}}\left(\frac{4}{3}\right)\right) \\ &= \ln\left(4\sqrt{\frac{8}{15}}\right) \\ &\approx \boxed{1.0720} \end{aligned}$$

Lesson 26: Lagrange Multipliers — Constrained Min/Max (II)

1. Solutions to In-Class Examples

Example 1. There is an ant on a circular heated plate which has a radius of 10 meters. Let x and y be the meters from the center of the plate measured horizontally and vertically respectively. Suppose the temperature of the plate is given by $f(x, y) = x^2 - y^2 + 150°$ F and that the ant is walking along the edge of the plate. What is the warmest spot the ant can find?

Solution: We want to maximize the function $f(x, y) = x^2 - y^2 + 150$ subject to the constraint $x^2 + y^2 = 100$ because the ant is walking around the *edge* of a plate with a radius of 10 meters. Hence, $g(x, y) = x^2 + y^2 = 100$. Differentiating,

$$f_x = 2x, \quad f_y = -2y, \quad g_x = 2x, \quad g_y = 2y.$$

So we set up our equations:

$$\boxed{\begin{aligned} 2x &= 2\lambda x \\ -2y &= 2\lambda y \\ x^2 + y^2 &= 100 \end{aligned}}$$

This is the system we need to solve.

From the first equation, we get

$$\begin{aligned} 2x &= 2\lambda x \\ \Rightarrow \quad x &= \lambda x \\ \Rightarrow \quad 0 &= \lambda x - x \\ \Rightarrow \quad 0 &= x(\lambda - 1) \end{aligned}$$

The first equation implies that either $x = 0$ or $\lambda = 1$. Similarly, the second equation implies

$$\begin{aligned} -2y &= 2\lambda y \\ \Rightarrow \quad y &= -\lambda y \\ \Rightarrow \quad y + \lambda y &= 0 \\ \Rightarrow \quad y(1 + \lambda) &= 0 \end{aligned}$$

which means either $y = 0$ or $\lambda = -1$.

Now, λ cannot be 1 and -1 at the same time, so we have a choice. If $\lambda = 1$, then $y = 0$. By our constraint, this implies $x = \pm 10$. So $(\pm 10, 0)$ are points we ought to check. If $\lambda = -1$, then $x = 0$. Thus, our constraint implies $y = \pm 10$. So the other points we need to check are $(0, \pm 10)$.

Putting this together, we have

$$f(10, 0) = (\pm 10)^2 - 0^2 + 150 = 100 + 150 = 250 \leftarrow \text{Max}$$
$$f(0, 10) = 0^2 - (\pm 10)^2 + 150 = 50 \leftarrow \text{Min.}$$

This means that our answer is $f(\pm 10, 0) = \boxed{250^\circ \text{ F}}$.

Example 2. A rectangular box with a square base is to be constructed from material that costs \$5/ft^2 for the bottom, \$4/ft^2 for the top, and \$10/ft^2 for the sides. Find the box of the greatest volume that can be constructed for \$216. Round your answer to 4 decimal places.

Solution: Since we are assuming the box has a square base, we see its volume is given by $V = w^2 h$ where w is the width and h is the height. Our cost function is then given by

$$C(w, h) = \underbrace{5w^2}_{\substack{\text{cost of} \\ \text{bottom}}} + \underbrace{4w^2}_{\substack{\text{cost of} \\ \text{top}}} + \underbrace{10(4wh)}_{\substack{\text{cost of} \\ \text{sides}}} = 9w^2 + 40wh.$$

Moreover, we are told that our cost will be \$216. Therefore, in this context our volume acts as our f and the cost function acts as our g. This is to say we are maximizing the volume subject to the constraint $C(w, h) = 216$.

Differentiating, we get

$$V_w = 2wh, \quad V_h = w^2, \quad C_w = 18w + 40h, \quad C_h = 40w.$$

Thus, the system we need to solve is

$$\boxed{\begin{aligned} 2wh &= \lambda(18w + 40h) \\ w^2 &= \lambda(40w) = 40\lambda w \\ 9w^2 + 40wh &= 216. \end{aligned}}$$

We note that because we need our box to have some volume, we must have $w \neq 0$ and $h \neq 0$. Given $w^2 = 40\lambda w$, we are able to divide through by w because we know it is nonzero. Therefore, $w = 40\lambda \Rightarrow \lambda = \dfrac{w}{40}$. Going to the first equation, we substitute:

$$\begin{aligned} 2wh &= \lambda(18w + 40h) \\ &= \frac{w}{40}(18w + 40h) \\ &= \frac{9}{20}w^2 + wh. \end{aligned}$$

Therefore, by subtracting wh from both sides we get $wh = \dfrac{9}{20}w^2$. Again, $w \neq 0$ so $h = \dfrac{9}{20}w$. Since we have a relationship between w and h that doesn't involve λ, we plug this back into our constraint to get

$$\begin{aligned} 216 &= 9w^2 + 40w \underbrace{\left(\frac{9}{20}w\right)}_{h} \\ &= 9w^2 + 18w^2 \\ &= 27w^2 \\ \Rightarrow \quad w^2 &= \frac{216}{27} \\ \Rightarrow \quad w &= \sqrt{\frac{216}{27}} = \sqrt{8} = 2\sqrt{2}. \end{aligned}$$

This means $h = \frac{9}{20}\underbrace{(2\sqrt{2})}_{w} = \frac{9\sqrt{2}}{10}$. Therefore, the point we need to check is $\left(2\sqrt{2}, \frac{9\sqrt{2}}{10}\right)$. The maximum volume is

$$V\left(2\sqrt{2}, \frac{9\sqrt{2}}{10}\right) = \left(2\sqrt{2}\right)^2\left(\frac{9\sqrt{2}}{10}\right) = \frac{72\sqrt{2}}{10} = \frac{36\sqrt{2}}{5} \approx \boxed{10.1823 \text{ ft}^3}.$$

EXAMPLE 3. A rectangular building with a square front is to be constructed of materials that cost \$10 per ft^2 for the flat roof, \$20 per ft^2 for the sides and back, and \$15 per ft^2 for the glass front. We will ignore the bottom of the building. If the volume of the building is 10,000 ft^3, what dimensions will minimize the cost of materials?

Solution: Observe that we are asked to find the *dimensions* which minimize the cost. We will use the same method but as an answer we need to state dimensions instead of a minimal cost.

Because we assume the building has a square front, we know that two of the dimensions are the same. So we can write $V = wh^2$ where w is the width and h is the height. Then, our cost function is given by

$$C(w,h) = \underbrace{10wh}_{\text{top}} + \underbrace{20(h^2 + 2wh)}_{\text{sides and back}} + \underbrace{15h^2}_{\text{front}}.$$

Simplifying, this becomes

$$C(w,h) = 35h^2 + 50wh.$$

Because this is subject to the constraint $V = wh^2 = 10{,}000$, we see that the cost acts as our f and the volume acts as our g, which is to say we are minimizing the cost subject to the constraint that the volume is 10,000 ft^3. Differentiating,

$$C_w = 50h, \quad C_h = 70h + 50w, \quad V_w = h^2, \quad V_h = 2wh.$$

Now, we the system we need to solve is

$$\boxed{\begin{aligned} 50h &= \lambda h^2 \\ 70h + 50w &= 2\lambda wh \\ wh^2 &= 10{,}000 \end{aligned}}$$

We note that because our volume is nonzero, $w \neq 0$ and $h \neq 0$. Thus, $50h = \lambda h^2 \Rightarrow 50 = \lambda h$. Moreover, $\lambda = \dfrac{50}{h}$. The second equation then implies

$$\begin{aligned} & 70h + 50w = 2\lambda wh \\ \Rightarrow \quad & 70h + 50w = 2\underset{\overset{\uparrow}{\lambda}}{\left(\frac{50}{h}\right)} wh = 100w \end{aligned}$$

Subtracting $50w$ from both sides, we get $70h = 50w \Rightarrow w = \dfrac{7}{5}h$. Returning to our constraint,

$$10{,}000 = wh^2 = \frac{7}{5}h(h^2) = \frac{7}{5}h^3 \quad \Rightarrow \quad \frac{50{,}000}{7} = h^3 \quad \Rightarrow \quad h = \sqrt[3]{\frac{50{,}000}{7}}.$$

So $w = \dfrac{7}{5}\sqrt[3]{\dfrac{50{,}000}{7}}$.

Thus, the dimensions that minimize the cost are

$$\boxed{w = \frac{7}{5}\sqrt[3]{\frac{50{,}000}{7}} \text{ and } h = \sqrt[3]{\frac{50{,}000}{7}}}.$$

Example 4. On a certain island, at any given time, there are R hundred rats and S hundred snakes. Their populations are related by the equation

$$(R - 16)^2 + 20(S - 16)^2 = 81.$$

What is the maximum combined number of rats and snakes that could ever be on the island at the same time? (Round your answer to the nearest integer).

Solution: Let $f(R, S) = R + S$ (which is the total number of rats and snakes in hundreds) and $g(R, S) = (R - 16)^2 + 20(S - 16)^2$ (which is our constraint function). Differentiating,

$$f_R = 1, \quad f_S = 1, \quad g_R = 2(R - 16), \quad g_S = 40(S - 16).$$

Then the system we need to solve is

$$\boxed{\begin{aligned} &1 = 2\lambda(R - 16) \\ &1 = 40\lambda(S - 16) \\ &(R - 16)^2 + 20(S - 16)^2 = 81. \end{aligned}}$$

The first two equations mean we can write

$$2\lambda(R - 16) = 1 = 40\lambda(S - 16).$$

This implies that $\lambda \neq 0$ and so dividing through by 2λ we get

$$R - 16 = 20(S - 16).$$

Therefore, returning to our constraint, we find

$$\begin{aligned} 81 &= (R-16)^2 + 20(S-16)^2 \\ &= (20(S-16))^2 + 20(S-16)^2 \\ &= 400(S-16)^2 + 20(S-16)^2 \\ &= 420(S-16)^2. \end{aligned}$$

Thus, $(S-16)^2 = \dfrac{81}{420} \Rightarrow S - 16 = \pm\dfrac{9}{\sqrt{420}}$. Further, $R - 16 = 20\left(\pm\dfrac{9}{\sqrt{420}}\right)$. Finally, we get

$$R = 20\left(\frac{9}{\sqrt{420}}\right) + 16 \approx 24.783 \text{ and } S = \frac{9}{\sqrt{420}} + 16 \approx 16.439$$

and

$$R = -20\left(\frac{9}{\sqrt{420}}\right) + 16 \approx 7.217 \text{ and } S = -\frac{9}{\sqrt{420}} + 16 \approx 15.561$$

Taking the larger R and larger S, the maximum combined number of rats and snakes will be

$$f\left(\underbrace{20\left(\frac{9}{\sqrt{420}}\right) + 16}_{R}, \underbrace{\frac{9}{\sqrt{420}} + 16}_{S}\right) = 20\left(\frac{9}{\sqrt{420}}\right) + 16 + \frac{9}{\sqrt{420}} + 16 \approx 41.2222$$

Since we are measuring in terms of hundreds, this means that we have 4122 rats and snakes.

2. Additional Examples

Examples.

1. Suppose Arjun has exactly 24 hours to study for an exam and, without preparation he will get 200 points out of 1000 total exam points. Suppose he estimates that his exam score will improve by $x(37-x)$ points if he reads lecture notes for x hours and $y(53-y)$ if he solves review problems for y hours. However, due to fatigue, he will lose $(x+y)^2$ points. What is the maximum score Arjun can obtain if he uses the entire 24 hours to study? Round your answer to the nearest hundredth.

Solution: We want to maximize Arjun's score, which is given by the function

$$\begin{aligned} f(x, y) &= 200 + x(37 - x) + y(53 - y) - (x + y)^2 \\ &= 200 + 37x - x^2 + 53y - y^2 - (x^2 + 2xy + y^2) \\ &= 200 + 37x - x^2 + 53y - y^2 - x^2 - 2xy - y^2 \\ &= 200 + 37x + 53y - 2xy - 2x^2 - 2y^2 \end{aligned}$$

Our constraint is $x + y = 24$ because we assume that Arjun uses the entire 24 hours to study. Our derivatives are

$$f_x = 37 - 2y - 4x, \quad f_y = 53 - 2x - 4y, \quad g_x = 1, \quad g_y = 1.$$

We set up our system of equations:

$$\begin{aligned} 37 - 2y - 4x &= \lambda \\ 53 - 2x - 4y &= \lambda \\ x + y &= 24 \end{aligned}$$

The first two equations imply that

$$\begin{aligned} 37 - 2y - 4x &= 53 - 2x - 4y \\ \Rightarrow \quad -2y &= 53 - 37 - 2x + 4x - 4y \\ \Rightarrow \quad -2y + 4y &= 16 + 2x \\ \Rightarrow \quad 2y &= 16 + 2x \\ \Rightarrow \quad y &= 8 + x \end{aligned}$$

Substituting this into the last equation,

$$\begin{aligned} 24 &= x + y \\ &= x + \underbrace{(8 + x)}_{y} \\ &= 2x + 8 \\ \Rightarrow \quad 24 &= 2x + 8 \\ \Rightarrow \quad 16 &= 2x \\ \Rightarrow \quad 8 &= x \end{aligned}$$

Then, because $y = 8 + x$, we have $y = 8 + 8 = 16$.

Our solution is $(8, 16)$.

Arjun's maximum score is then given by

$$\begin{aligned} f(8, 16) &= 200 + 8(37 - 8) + 16(53 - 16) - (8 + 16)^2 \\ &= 200 + 8(29) + 16(37) - (24)^2 \\ &= \boxed{448 \text{ points}} \end{aligned}$$

2. Suppose an artist sells sketches of her cat and dog online and that she notices she can make a profit of

$$p(x,y) = x^{3/2}y^{1/2} \text{ dollars/day}$$

if she offers x sketches of her cat and y sketches of her dog at the beginning of a day. If she is only able to create 32 total sketches per day, what is the maximum profit she can make per day? Round your answer to the nearest cent.

Solution: We want to maximize $p(x,y) = x^{3/2}y^{1/2}$ subject to $g(x,y) = x + y = 32$. Our derivatives are

$$p_x = \frac{3}{2}x^{3/2-1}y^{1/2} = \frac{3}{2}x^{1/2}y^{1/2}, \quad p_y = \frac{1}{2}x^{3/2}y^{1/2-1} = \frac{1}{2}x^{3/2}y^{-1/2}, \quad g_x = 1, \quad g_y = 1.$$

Our system of equations is

$$\begin{aligned} \frac{3}{2}x^{1/2}y^{1/2} &= \lambda \\ \frac{1}{2}x^{3/2}y^{-1/2} &= \lambda \\ x + y &= 32 \end{aligned}$$

We see the first and second equations are both equal to λ and so we may write

$$\begin{aligned} \frac{3}{2}x^{1/2}y^{1/2} &= \frac{1}{2}x^{3/2}y^{-1/2} \\ \Rightarrow \quad 3x^{1/2}y^{1/2} &= x^{3/2}y^{-1/2} \\ \Rightarrow \quad 3x^{1/2}y^{1/2} - x^{3/2}y^{-1/2} &= 0 \\ \Rightarrow \quad x^{1/2}y^{1/2}(3 - xy^{-1}) &= 0 \end{aligned}$$

Now, we have three options: $x = 0$ **or** $y = 0$ **or** $3 - xy^{-1} = 0$.

Case 1: $x = 0$

If $x = 0$, then $y = 32$ and so one solution is $(0, 32)$.

Case 2: $y = 0$

If $y = 0$, then $x = 32$ and so another solution is $(32, 0)$.

Case 3: $3 - xy^{-1} = 0$

If $3 - xy^{-1} = 0$, then $3 = xy^{-1} \Rightarrow 3y = x$. Since $x + y = 32$, we write

$$\begin{aligned} 32 &= x + y \\ \Rightarrow \quad 32 &= 3y + y \\ \Rightarrow \quad 32 &= 4y \\ \Rightarrow \quad 8 &= y \end{aligned}$$

Thus, $x = 3y = 3(8) = 24$. Our last solution is $(24, 8)$.

We check the function values at each solutions:

$$f(0,32) = (0)^{3/2}(32)^{1/2} = 0 \leftarrow \text{Min}$$

$$f(32,0) = (32)^{3/2}(0)^{1/2} = 0 \leftarrow \text{Min}$$

$$f(24,8) = (24)^{3/2}(8)^{1/2} \approx \boxed{332.55} \leftarrow \text{Max}$$

Lesson 27: Double Integrals, Volume, Applications

1. Integrating Functions of Several Variables

In this lesson, we address how we integrate functions of several variables. The motivation is initially very geometric: we want to determine how much volume there is under the curve (just as we think of integrals as the area under a curve).

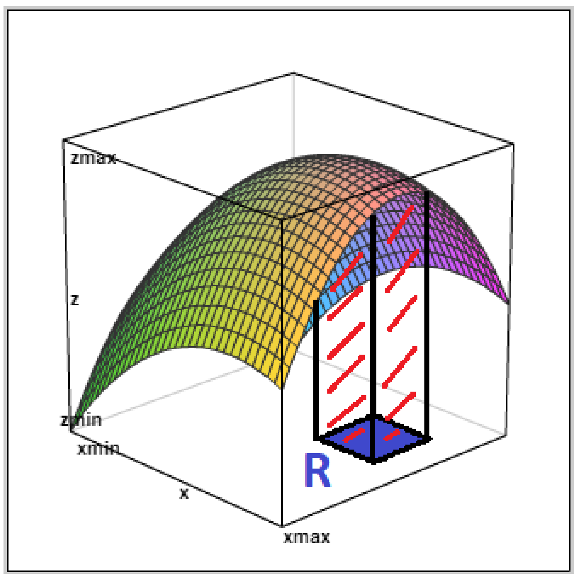


FIGURE 10. The volume under the function over the region R (blue square) is contained in the black and red box.

NOTE 66. The regions over which we integrate will always be **2**-dimensional.

The volume under $f(x, y)$ over the region R is denoted

$$\iint_R f(x, y)\, dA. \tag{16}$$

The dA simply stands for the area of the region R.

Now, although $\iint_R f(x, y)\, dA$ represents the volume under f over the region R, this is not the practical method by which we integrate. Instead, we need to carefully dissect the way R is described in terms of x and y.

Ex 1. Suppose R is given by

$$\begin{cases} 0 \leq x \leq 3 \\ -1 \leq y \leq 1 \end{cases}$$

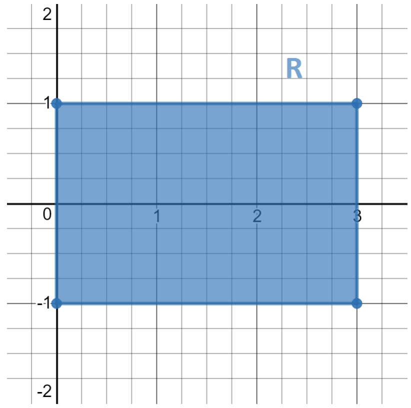


FIGURE 11. $0 \le x \le 3, \quad -1 \le y \le 1$

Although it doesn't matter for a rectangle, we interpret the description of R as:

$$\begin{cases} 0 \;\le\; x \;\le\; 3 & \longleftarrow \text{Dependent variable} \\ -1 \;\le\; y \;\le\; 1 & \longleftarrow \text{Independent variable} \end{cases}$$

Double integrals are **always** written with the dependent variable on the *inside* and the independent variable on the *outside*, that is,

$$\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_0^3 f(x,y)\,dx\,dy.$$

We **pair** the x-bounds on the inside with dx and we **pair** the y-bounds on the outside with dy. This is what is meant by pairing: the inside most integral and inside most differential go together and the outside most integral and outside most differential go together.

EX 2. Consider the function $f(x,y) = x + y$ and the region R bounded by

$$\begin{cases} 0 \;\le\; x \;\le\; 1 & \longleftarrow \text{Dependent variable} \\ 0 \;\le\; y \;\le\; 2 & \longleftarrow \text{Independent variable} \end{cases}$$

The volume under $f(x,y)$ over R is given by

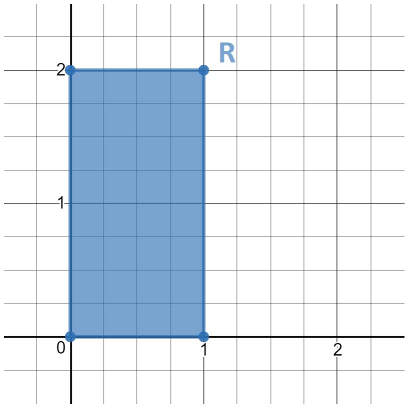
$$\iint_R f(x,y)\,dA = \int_0^2\int_0^1 (x+y)\,dx\,dy.$$

Observe here that the **pairing** of the integrals with the dx and dy is very important. Although the integral

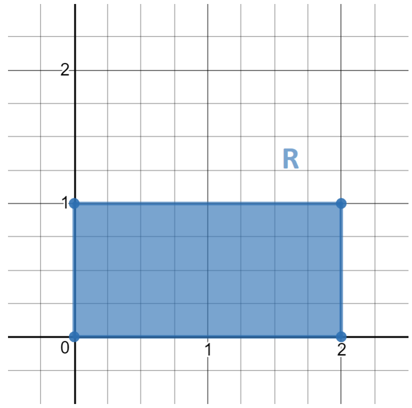
$$\int_0^1\int_0^2 (x+y)\,dx\,dy \tag{17}$$

might look very similar to integral (16) above, they are actually over different regions.

Now, we address how we actually compute a double integral.



$$\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 2 \end{cases}$$



$$\begin{cases} 0 \leq x \leq 2 \\ 0 \leq y \leq 1 \end{cases}$$

Ex 3. Compute

(a) $\displaystyle\int_0^2 \int_0^1 x\,dx\,dy$

We compute a double integral from the "inside" to the "outside". That is, we first look at the $d\cdot$ which is closest to the function.

For this integral, because the dx is the most inside, we integrate with respect to x first and consider all functions of y as a *constant* (just like we did with partial derivatives).

$$\int_0^2 \underbrace{\int_0^1 x\,dx}_{\text{compute first}}\,dy = \int_0^2 \left(\frac{1}{2}x^2 \Big|_{x=0}^{x=1} \right) dy$$

$$= \int_0^2 \left[\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 \right] dy$$

$$= \int_0^2 \frac{1}{2}\,dy$$

$$= \frac{1}{2}y \Big|_{y=0}^{y=2}$$

$$= \frac{1}{2}(2) - \frac{1}{2}(0) = \boxed{1}.$$

(b) $\displaystyle\int_0^2 \int_0^1 x\,dy\,dx$

Because the dy is the most inside, we integrate with respect to y first but this time we hold all functions of x as *constants*.

$$
\begin{aligned}
\int_0^2 \int_0^1 x\,dy\,dx &= \int_0^2 x \underbrace{\int_0^1 1\,dy}_{y}\,dx \text{ since } x \text{ is a constant with respect to } y \\
&= \int_0^2 xy\Big|_{y=0}^{y=1} dx \\
&= \int_0^2 [x\cdot(1) - x\cdot(0)]\,dx \\
&= \int_0^2 x\,dx \\
&= \frac{1}{2}x^2\Big|_{x=0}^{x=2} \\
&= \frac{1}{2}(2)^2 - \frac{1}{2}(0)^2 \\
&= \frac{1}{2}(4) = \boxed{2}
\end{aligned}
$$

NOTE 67. In the third line of this calculation, we replaced **only** the y-values leaving the functions of x alone.

This example demonstrates that the pairing is very important. However, if you keep the pairings consistent, you can swap the *order* in which you integrate.

$$
\int_0^2 \int_0^1 x\,dx\,dy = \int_0^1 \int_0^2 x\,dy\,dx
$$
$$
\int_0^1 \int_0^2 x\,dx\,dy = \int_0^2 \int_0^1 x\,dy\,dx
$$

EXAMPLES.

1. Integrate $\displaystyle\int_{-1}^1 \int_0^1 (2x + 6y)\,dy\,dx$

Solution:

$$\int_{-1}^{1}\int_{0}^{1}(2x+6y)\,dy\,dx = \int_{-1}^{1}(2xy+3y^2)\Big|_{y=0}^{y=1}\,dx$$

$$= \int_{-1}^{1}\left[(2x(1)+3(1)^2)-(2x(0)+3(0)^2)\right]\,dx$$

$$= \int_{-1}^{1}(2x+3)\,dx$$

$$= x^2+3x\Big|_{x=-1}^{x=1}$$

$$= ((1)^2+3(1))-((-1)^2+3(-1)) = \boxed{6}$$

2. Integrate $\displaystyle\int_{0}^{\pi/2}\int_{0}^{1}3y^2\cos x\,dy\,dx$

Solution:

$$\int_{0}^{\pi/2}\int_{0}^{1}3y^2\cos x\,dy\,dx = \int_{0}^{\pi/2}y^3\cos x\Big|_{y=0}^{y=1}\,dx$$

$$= \int_{0}^{\pi/2}\left[(1)^3\cos x-(0)^3\cos x\right]\,dx$$

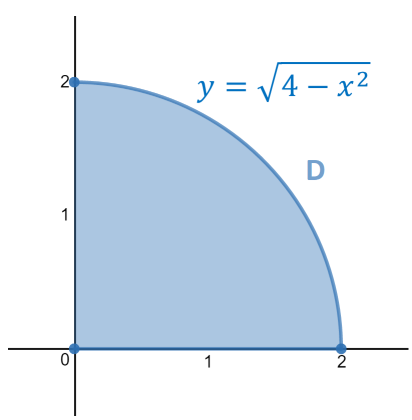
$$= \int_{0}^{\pi/2}\cos x\,dx$$

$$= \sin x\Big|_{x=0}^{x=\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right)-\sin(0) = \boxed{1}.$$

We can integrate over more than rectangles as long as we can describe the region in an appropriate way.

Ex 4. Integrate the function $f(x,y)=2y$ over the region D where



This region is described by

$$\begin{cases} 0 \leq y \leq \sqrt{4-x^2} & \longleftarrow \text{Dependent variable} \\ 0 \leq x \leq 2 & \longleftarrow \text{Independent variable} \end{cases}$$

Our integral is then denoted by

$$\int_0^2 \int_0^{\sqrt{4-x^2}} 2y \, dy \, dx.$$

NOTE 68. **DO NOT WRITE**

$$\int_0^{\sqrt{4-x^2}} \int_0^2 2y \, dx \, dy.$$

Here, the **dependent variable** is on the *outside*, which is **no** good. For this homework, the outside integral can only have numbers for its bounds.

We compute the integral:

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{4-x^2}} 2y \, dy \, dx &= \int_0^2 y^2 \Big|_{y=0}^{y=\sqrt{4-x^2}} dx \\ &= \int_0^2 \left[\left(\sqrt{4-x^2} \right)^2 - (0)^2 \right] dx \\ &= \int_0^2 (4 - x^2) \, dx \\ &= 4x - \frac{1}{3}x^3 \Big|_{x=0}^{x=2} \\ &= \left[4(2) - \frac{1}{3}(2)^3 \right] - \left[4(0) - \frac{1}{3}(0)^3 \right] \\ &= 8 - \frac{8}{3} = \boxed{\frac{16}{3}} \end{aligned}$$

When the region of integration is not a square, you can **only** swap the order of integration if you can describe the region such that the dependent variable becomes the independent variable and vice versa. In fact, sometimes you must swap the order of integration to make the integral possible to compute. This will be addressed in the next lesson.

EXAMPLES.

3. Integrate $\int_1^e \int_0^{\ln x} x \, dy \, dx$

Solution:

$$\int_1^e \int_0^{\ln x} x\,dy\,dx = \int_1^e xy\Big|_{y=0}^{y=\ln x} dx$$

$$= \int_1^e [x(\ln x) - x(0)]\,dx$$

$$= \int_1^e [x\ln x]\,dx$$

This is now an integration by parts problem.

By LIATE, our table is

$$u = \ln x \qquad dv = x\,dx$$

$$du = \frac{1}{x}\,dx \qquad v = \frac{1}{2}x^2$$

Thus,

$$\int_1^e x\ln x\,dx = \frac{1}{2}x^2\ln x\Big|_{x=1}^{x=e} - \int_1^e \frac{1}{2}x\,dx$$

$$= \frac{1}{2}x^2\ln x\Big|_{x=1}^{x=e} - \frac{1}{2}\left(\frac{1}{2}\right)x^2\Big|_{x=1}^{x=e}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2\Big|_{x=1}^{x=e}$$

$$= \frac{1}{2}(e)^2\underbrace{\ln(e)}_{1} - \frac{1}{4}(e)^2 - \left[\frac{1}{2}(1)^2\underbrace{\ln(1)}_{0} - \frac{1}{4}(1)^2\right]$$

$$= \frac{1}{2}e^2 - \frac{1}{4}(e^2 - 1)$$

$$= \boxed{\frac{1}{4}e^2 + \frac{1}{4}}$$

4. Compute $\int_0^{\pi/2}\int_y^{\pi/2}(-\sec(y)\sin(x))\,dx\,dy$ **Solution**:

$$\int_0^{\pi/2}\int_y^{\pi/2}(-\sec(y)\sin(x))\,dx\,dy = \int_0^{\pi/2}(\sec(y)\cos(x))\Big|_{x=y}^{x=\pi/2} dy$$

$$= \int_0^{\pi/2}\left[\sec(y)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} - \sec(y)\cos(y)\right] dy$$

$$= \int_0^{\pi/2} -\frac{1}{\cos(y)}\cos(y)\,dy$$

$$= \int_0^{\pi/2} (-1)\, dy$$

$$= -y \Big|_{y=0}^{y=\pi/2}$$

$$= \boxed{-\frac{\pi}{2}}$$

2. Additional Example

Examples.

1. Evaluate $\displaystyle\int_3^6 \int_0^y 10xy\, dx\, dy$.

Solution: Write

$$\int_3^6 \int_0^y 10xy\, dx\, dy = \int_3^6 5x^2y \Big|_{x=0}^{x=y} dy$$

$$= \int_3^6 [5(y)^2 y - 5(0)^2 y]\, dy$$

$$= \int_3^6 5y^3\, dy$$

$$= \frac{5}{4} y^4 \Big|_{y=3}^{y=6}$$

$$= \frac{5}{4}(6)^4 - \frac{5}{4}(3)^4$$

$$= \boxed{\frac{6075}{4}}$$

2. Evaluate $\displaystyle\int_4^7 \int_3^x \frac{5x^2}{y^2}\, dy\, dx$.

Solution: Write

$$\int_4^7 \int_3^x \frac{5x^2}{y^2}\, dy\, dx = \int_4^7 \int_3^x 5x^2 y^{-2}\, dy\, dx$$

$$= \int_4^7 -5x^2 y^{-1} \Big|_{y=3}^{y=x} dx$$

$$= \int_4^7 \left[-5x^2 (x)^{-1} + 5x^2 (3)^{-1}\right]\, dx$$

$$= \int_4^7 \left[-5x + \frac{5}{3}x^2\right] dx$$

$$= -\frac{5}{2}x^2 + \frac{5}{9}x^3 \Big|_{x=4}^{x=7}$$

$$= -\frac{5}{2}(7)^2 + \frac{5}{9}(7)^3 - \left[-\frac{5}{2}(4)^2 + \frac{5}{9}(4)^3\right]$$

$$= \boxed{\frac{145}{2}}$$

3. Evaluate $\int_0^{\sqrt{\pi/3}} \int_{x^2}^{\pi/2} 5x \sin y \, dy \, dx$.

Solution: Write

$$\int_0^{\sqrt{\pi/3}} \int_{x^2}^{\pi/2} 5x \sin y \, dy \, dx = \int_0^{\sqrt{\pi/3}} -5x \cos y \Big|_{y=x^2}^{x=\pi/2} dx$$

$$= \int_0^{\sqrt{\pi/3}} \left[-5x \cos(x^2) - \left(-5x \cos\left(\frac{\pi}{2}\right)\right)\right] dx$$

$$= \int_0^{\sqrt{\pi/3}} 5x \cos(x^2) dx$$

$$= \frac{5}{2} \sin(x^2)\Big|_0^{\sqrt{\pi/3}}$$

$$= \frac{5}{2} \sin\left(\sqrt{\frac{\pi}{3}}\right)^2 - \frac{5}{2}\sin(0)^2$$

$$= \frac{5}{2} \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{5}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{5\sqrt{3}}{4}}$$

Lesson 28: Double Integrals, Volume, Applications (II)

1. Double Integrals

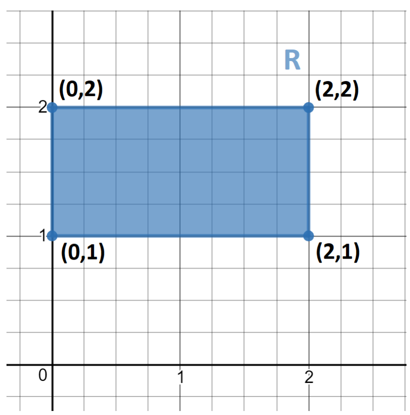
We continue working with double integrals and begin integrating over more regions than just rectangles and practice swapping the order of integration.

EXAMPLES.

1. Suppose R is a rectangle with vertices $(0,1),(0,2),(2,1),(2,2)$. Find

$$\iint_R 4x^3y\,dA.$$

Solution: We sketch a picture of our region



and note that this is described by

$$\begin{cases} 1 \leq y \leq 2 \\ 0 \leq x \leq 2 \end{cases}$$

Thus,

$$\begin{aligned}\iint_R 4x^3y\,dA &= \int_0^2 \int_1^2 4x^3y\,dy\,dx \\ &= \int_0^2 2x^3y^2\Big|_{y=1}^{y=2}\,dx \\ &= \int_0^2 2x^3[(2)^2-(1)^2]\,dx\end{aligned}$$

$$= \int_0^2 6x^3\,dx$$

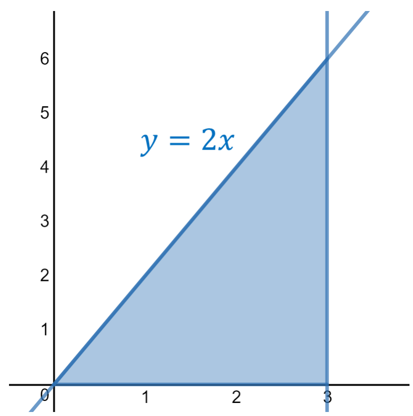
$$= \frac{3}{2}x^4\Big|_{x=0}^{x=2}$$

$$= \frac{3}{2}\left[(2)^4 - (0)^4\right] = \boxed{24}$$

2. Suppose R is the region bounded by the x-axis, $y = 2x$, and $x = 3$. Find

$$\iint_R (x+y)\,dA.$$

Solution: Again, we sketch a picture of our region.



Our region is described by

$$\begin{cases} 0 \le y \le 2x \\ 0 \le x \le 3 \end{cases}$$

Hence,

$$\iint_R (x+y)\,dA = \int_0^3 \int_0^{2x} (x+y)\,dy\,dx$$

$$= \int_0^3 \left[xy + \frac{1}{2}y^2\right]_{y=0}^{y=2x} dx$$

$$= \int_0^3 \left[x(2x) + \frac{1}{2}(2x)^2\right] dx$$

$$= \int_0^3 \left[2x^2 + \frac{1}{2}(4x^2)\right] dx$$

$$= \int_0^3 4x^2\,dx$$

$$= \frac{4}{3}x^3\Big|_{x=0}^{x=3}$$

$$= \frac{4}{3}(3)^3 - \frac{4}{3}(0)^3 = \boxed{36}$$

3. Find the volume below $z = 5 + 10y$ above the region R bounded by $-5 \leq x \leq 5$ and $0 \leq y \leq 25 - x^2$.

Solution: The volume below $z = 5+10y$ above a particular R is exactly $\iint_R (5 + 10y)\, dA$. The region is described by

$$\begin{cases} 0 \leq y \leq 25 - x^2 \\ -5 \leq x \leq 5 \end{cases}$$

Hence,

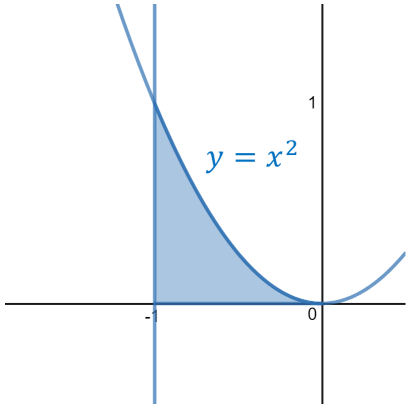
$$\begin{aligned}
\iint_R (5 + 10y)\, dA &= \int_{-5}^{5}\int_0^{25-x^2} (5 + 10y)\, dy\, dx \\
&= \int_{-5}^{5} (5y + 5y^2)\Big|_{y=0}^{y=25-x^2} dx \\
&= \int_{-5}^{5} 5\left[(25 - x^2) + (25 - x^2)^2\right] dx \\
&= \int_{-5}^{5} 5(x^4 - 51x^2 + 650)\, dx \\
&= 5\left[\frac{1}{5}x^5 - \frac{51}{3}x^3 + 650x\right]_{x=-5}^{x=5} \\
&= 5\left[\frac{1}{5}(5)^5 - \frac{51}{3}(5)^3 + 650(5) - \left(\frac{1}{5}(-5)^5 - \frac{51}{3}(-5)^3 + 650(-5)\right)\right] \\
&= \boxed{17{,}500}
\end{aligned}$$

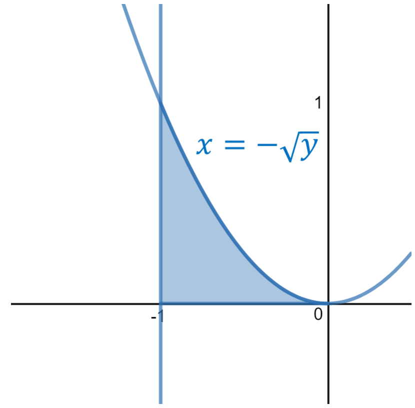
4. Given

$$\int_{-1}^{0}\int_0^{x^2} f(x, y)\, dy\, dx,$$

swap the order of integration.

Solution: The point of this problem is to emphasize that swapping the order of integration does not depend on the function we are integrating — only the bounds. Here, we need to sketch graph of the region described by the bounds and observe that this region can be described in two different ways.





$$0 \le y \le x^2 \qquad -1 \le x \le -\sqrt{y}$$

$$-1 \le x \le 0 \qquad 0 \le y \le 1$$

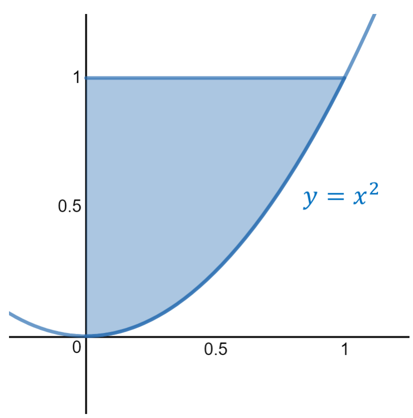
The problem describes the region as in the first picture. To switch the order of integration, we use the description in the second picture. Thus,

$$\int_{-1}^{0}\int_{0}^{x^2} f(x,y)\,dy\,dx = \int_{0}^{1}\int_{-1}^{-\sqrt{y}} f(x,y)\,dx\,dy.$$

5. Evaluate

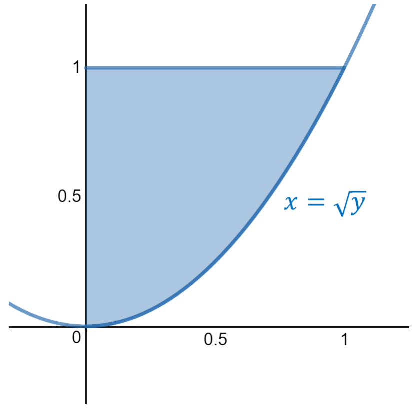
$$\int_0^1\int_{x^2}^1 2x\sqrt{1+y^2}\,dy\,dx.$$

Solution: To compute this integral, we will need to switch the order of integration because we do not have the integration tools to address this problem as written. As above, we sketch the region:



Changing our bounds so that x is a function of y, we see that our region is also described by

$$\begin{cases} 0 \le x \le \sqrt{y} \\ 0 \le y \le 1 \end{cases}$$



Thus,

$$\begin{aligned}
\int_0^1 \int_{x^2}^1 2x\sqrt{1+y^2}\, dy\, dx &= \int_0^1 \int_0^{\sqrt{y}} 2x\sqrt{1+y^2}\, dx\, dy \\
&= \int_0^1 x^2\sqrt{1+y^2}\Big|_{x=0}^{x=\sqrt{y}} dy \\
&= \int_0^1 \left[(\sqrt{y})^2\sqrt{1+y^2} - (0)^2\sqrt{1+y^2}\right] dy \\
&= \int_0^1 y\sqrt{1+y^2}\, dy
\end{aligned}$$

Now, the integral becomes a u-substitution problem.

Let $u = 1+y^2$, then $du = 2y\,dy$, $u(0) = 1+0^2 = 1$, and $u(1) = 1+1^2 = 2$. So,

$$\begin{aligned}
\int_0^1 y\sqrt{1+y^2}\, dy &= \frac{1}{2}\int_1^2 \sqrt{u}\, du \\
&= \frac{1}{2}\left(\frac{2}{3}\right) u^{3/2}\Big|_1^2 \\
&= \boxed{\frac{1}{3}(2^{3/2}-1)}.
\end{aligned}$$

6. Let R be the region bounded by the x-axis, $y = \sin x$, $x = \dfrac{\pi}{6}$, and $x = \dfrac{\pi}{3}$. Evaluate $\displaystyle\iint_R \sec^2(x)\, dy\, dx$.

Solution: Since y is a function of x, that needs to be the inside integral (this also follows by how the problem is presented). Further, we won't want to switch the order of integration because that will make our bounds more complicated. We have

$$\begin{cases} 0 \leq y \leq \sin x \\ \dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{3} \end{cases}$$

We write

$$\begin{aligned}
\iint_R \sec^2(x)\, dy\, dx &= \int_{\pi/6}^{\pi/3} \int_0^{\sin x} \sec^2(x)\, dy\, dx \\
&= \int_{\pi/6}^{\pi/3} y \sec^2(x) \Big|_{y=0}^{y=\sin x} dx \\
&= \int_{\pi/6}^{\pi/3} (\sin(x)) \sec^2(x) dx \\
&= \int_{\pi/6}^{\pi/3} \frac{\sin(x)}{\cos^2(x)}\, dx
\end{aligned}$$

This is another u-substitution problem. Let $u = \cos(x)$, then $du = -\sin(x)\, dx$, $u\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, and $u\left(\frac{\pi}{3}\right) = \frac{1}{2}$. So,

$$\begin{aligned}
\int_{\pi/6}^{\pi/3} \frac{\sin(x)}{\cos^2(x)}\, dx &= \int_{\sqrt{3}/2}^{1/2} -\frac{1}{u^2}\, du \\
&= \frac{1}{u}\Big|_{\sqrt{3}/2}^{1/2} \\
&= \boxed{2 - \frac{2}{\sqrt{3}}}
\end{aligned}$$

2. Additional Examples

EXAMPLES.

1. Evaluate $\displaystyle\int_0^8 \int_{\sqrt{y}/3}^{\sqrt{y}} x\sqrt{64 - y^2}\, dx\, dy$.

Solution: Write

$$\int_0^8 \int_{\sqrt{y}/3}^{\sqrt{y}} x\sqrt{64-y^2}\,dx\,dy = \int_0^8 \frac{1}{2}x^2\sqrt{64-y^2}\Big|_{x=\sqrt{y}/3}^{x=\sqrt{y}} dy$$

$$= \int_0^8 \left[\frac{1}{2}(\sqrt{y})^2\sqrt{64-y^2} - \frac{1}{2}\left(\frac{\sqrt{y}}{3}\right)^2\sqrt{64-y^2}\right] dy$$

$$= \int_0^8 \left[\frac{y}{2}\sqrt{64-y^2} - \frac{y}{18}\sqrt{64-y^2}\right] dy$$

$$= \int_0^8 \frac{4}{9}y\sqrt{64-y^2}\,dy$$

This is now a u-substitution problem. Let $u = 64 - y^2$, then $du = -2y\,dy$ which implies $dy = \dfrac{du}{-2y}$. So we write

$$\int_0^8 \frac{4}{9}y\sqrt{64-y^2}\,dy = \int_{u(0)}^{u(8)} \frac{4}{9}y\sqrt{u}\underbrace{\left(\frac{du}{-2y}\right)}_{dy}$$

$$= \int_{u(0)}^{u(8)} -\frac{2}{9}u^{1/2}\,du$$

$$= -\frac{2}{9}\left(\frac{1}{1/2+1}\right)u^{1/2+1}\Big|_{u(0)}^{u(8)}$$

$$= -\frac{2}{9}\left(\frac{1}{3/2}\right)u^{3/2}\Big|_{u(0)}^{u(8)}$$

$$= -\frac{2}{9}\left(\frac{2}{3}\right)u^{3/2}\Big|_{u(0)}^{u(8)}$$

$$= -\frac{4}{27}(64-y^2)^{3/2}\Big|_0^8$$

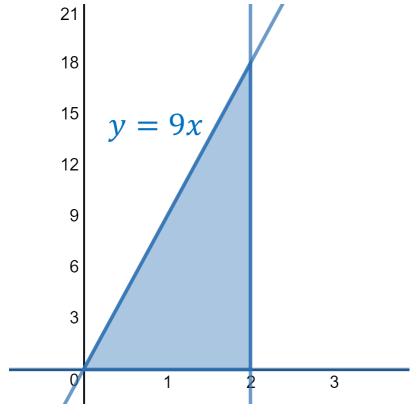
$$= -\frac{4}{27}(64-(8)^2)^{3/2} + \frac{4}{27}(64-(0)^2)^{3/2}$$

$$= \frac{4}{27}(64)^{3/2} = \boxed{\frac{2048}{27}}$$

2. Evaluate $\displaystyle\iint_R \frac{1}{x^2+8}\,dA$ where R is the region bounded by

$$y = 9x, \quad x\text{-axis}, \quad x = 2.$$

Solution: We sketch a quick picture of our region:



Before we describe this region, observe that we need to integrate with respect to y first because we can't integrate

$$\int \frac{1}{x^2+8}\,dx$$

with our current integration techniques. This means that we need to describe the region such that y is the dependent variable and x is the independent variable. We see that our x-values will vary between 0 and 2 and our y-values can never be larger than the curve $y = 9x$. So we write

$$\begin{cases} 0 \le y \le 9x \\ 0 \le x \le 2 \end{cases}$$

Hence, our integral becomes

$$\begin{aligned} \iint_R \frac{1}{x^2+8}\,dA &= \int_0^2 \int_0^{9x} \frac{1}{x^2+8}\,dy\,dx \\ &= \int_0^2 \left.\frac{y}{x^2+8}\right|_{y=0}^{y=9x} dx \\ &= \int_0^2 \frac{9x}{x^2+8}\,dx \end{aligned}$$

Next, we take $u = x^2+8$, then $du = 2x\,dx \;\Rightarrow\; dx = \dfrac{du}{2x}$. Write

$$\begin{aligned} \int_0^2 \frac{9x}{x^2+8}\,dx &= \int_{u(0)}^{u(2)} \frac{9x}{u}\underbrace{\left(\frac{du}{2x}\right)}_{dx} \\ &= \int_{u(0)}^{u(2)} \frac{9}{2u}\,du \end{aligned}$$

$$= \frac{9}{2} \ln |u| \Big|_{u(0)}^{u(2)}$$

$$= \frac{9}{2} \ln |x^2 + 8| \Big|_{0}^{2}$$

$$= \frac{9}{2}(\ln((2)^2 + 8) - \ln((0)^2 + 8))$$

$$= \frac{9}{2}(\ln(12) - \ln(8))$$

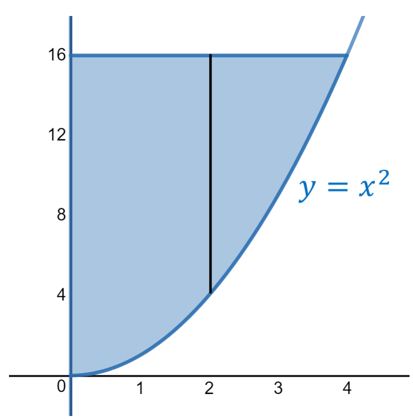
$$= \boxed{\frac{9}{2} \ln \left(\frac{3}{2}\right)}$$

3. Evaluate $\displaystyle\int_0^4 \int_{x^2}^{16} 2x\sqrt{1 + y^2}\, dy\, dx$.

Solution: We cannot integrate in the given order (with our techniques), so we need to swap the order of integration. The region described is

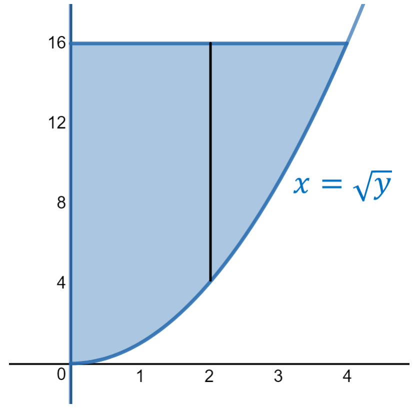
$$\begin{cases} x^2 \leq y \leq 16 \\ 0 \leq x \leq 4 \end{cases}$$

where y is the dependent variable and x is the independent variable. Our region looks like



Now, we need y to be the independent variable, which means we need to solve for x. Write

$$y = x^2 \quad \Rightarrow \quad x = \sqrt{y}.$$



Next, we observe that y varies between 0 and 16 and that x must always be *less than* the curve $x = \sqrt{y}$. Hence, our new description is

$$\begin{cases} 0 \leq x \leq \sqrt{y} \\ 0 \leq y \leq 16 \end{cases}$$

We write

$$\int_0^{16}\int_0^{\sqrt{y}} 2x\sqrt{1+y^2}\,dx\,dy = \int_0^{16} x^2\sqrt{1+y^2}\Big|_{x=0}^{x=\sqrt{y}}\,dy$$

$$= \int_0^{16} (\sqrt{y})^2\sqrt{1+y^2}\,dy$$

$$= \int_0^{16} y\sqrt{1+y^2}\,dy$$

Now, we use u-substitution. Let $u = 1+y^2$, then $du = 2y\,dy \Rightarrow dy = \dfrac{du}{2y}$. Further,

$$u(0) = 1+(0)^2 = 1$$

$$u(16) = 1+(16)^2 = 257$$

We write

$$\int_0^{16} y\sqrt{1+y^2}\,dy = \int_1^{257} y\sqrt{u}\,\underbrace{\left(\frac{du}{2y}\right)}_{dy}$$

$$= \int_1^{257} \frac{1}{2}u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{1}{1/2+1}\right)u^{1/2+1}\Big|_{u=1}^{u=257}$$

$$= \frac{1}{2}\left(\frac{1}{3/2}\right)u^{3/2}\Big|_{1}^{257}$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)u^{3/2}\Big|_{1}^{257}$$

$$= \frac{1}{3}\left[(257)^{3/2} - (1)^{3/2}\right]$$

$$= \boxed{\frac{1}{3}\left[(257)^{3/2} - 1\right]}$$

Lesson 29: Double Integrals, Volume, Applications (III)

1. Double Integral Examples

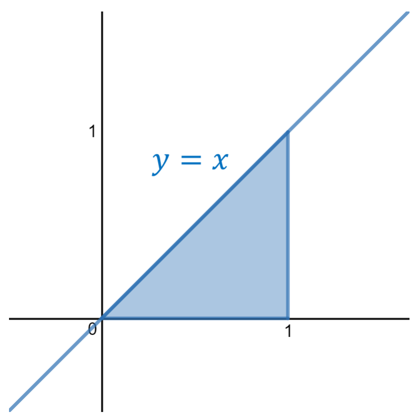
We continue working with double integrals and continue to practice swapping the order of integration. We also address how to find the average of a function over a rectangle.

Examples.

1. Evaluate

$$\int_0^1 \int_y^1 e^{x^2} \, dx \, dy.$$

Solution: We must swap the order of integration because e^{x^2} cannot be integrated. Our region ($y \leq x \leq 1$, $0 \leq y \leq 1$) is graphed as



Note that our region can also be described by

$$\begin{cases} 0 \leq y \leq x \\ 0 \leq x \leq 1 \end{cases}$$

Therefore, we may write

$$\begin{aligned} \int_0^1 \int_y^1 e^{x^2} \, dx \, dy &= \int_0^1 \int_0^x e^{x^2} \, dy \, dx \\ &= \int_0^1 y e^{x^2} \Big|_{y=0}^{y=x} \, dx \end{aligned}$$

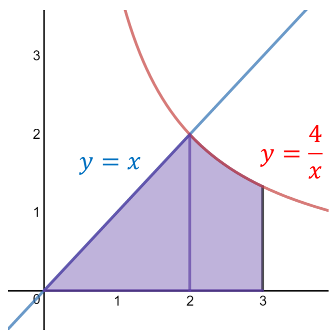
$$= \int_0^1 xe^{x^2}\,dx$$

This is a u-substitution problem. Let $u = x^2$, then $du = 2x\,dx$, $u(0) = 0^2 = 0$, and $u(1) = 1^2 = 1$. So

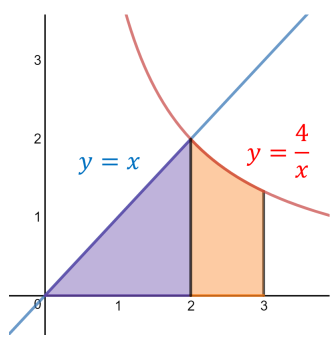
$$\int_0^1 xe^{x^2}\,dx = \int_0^1 \frac{1}{2}e^u\,du = \frac{1}{2}e^u\Big|_0^1 = \boxed{\frac{1}{2}(e-1)}$$

2. Find $\displaystyle\iint_R x^2\,dA$ where R is the region in the first quadrant bounded by $xy = 4$, $y = x$, $y = 0$, and $x = 3$.

Solution: From $xy = 4$, we get $y = \dfrac{4}{x}$ which we can graph. Here, it is particularly important to sketch a graph of the function else we will miss an important observation.



To integrate, we need to describe this region. However, there is more than one function at play. We address this by splitting this region into two regions depending on which function is enclosing the area.



Region 1: $\begin{array}{ccccc} 0 & \le & y & \le & x \\ 0 & \le & x & \le & 2 \end{array}$

Region 2: $\begin{array}{ccccc} 0 & \le & y & \le & \dfrac{4}{x} \\ 2 & \le & x & \le & 3 \end{array}$

We write

$$\begin{aligned}\iint_R x^2\,dA &= \underbrace{\int_0^2\int_0^x x^2\,dy\,dx}_{\text{Region 1}} + \underbrace{\int_2^3\int_0^{4/x} x^2\,dy\,dx}_{\text{Region 2}}\\ &= \int_0^2 x^2y\Big|_{y=0}^{y=x}\,dx + \int_2^3 x^2y\Big|_{y=0}^{y=4/x}\,dx\\ &= \int_0^2 x^3\,dx + \int_2^3 4x\,dx\\ &= \frac{1}{4}x^4\Big|_{x=0}^{x=2} + 2x^2\Big|_{x=2}^{x=3}\\ &= \frac{1}{4}(2)^4 + \big(2(2)^3 - 2(2)^2\big)\\ &= 4 + 18 - 8 = \boxed{14}.\end{aligned}$$

2. Average Value of Functions of Several Variables

The **average value** of a function $f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ is given by

$$\text{Ave}_f = \frac{1}{(b-a)(d-c)}\iint_R f(x,y)\,dA = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dA.$$

EXAMPLES.

3. Find the average value of $f(x,y) = \dfrac{15\ln 2}{4}e^y\sqrt{x+e^y}$ over the rectangle with vertices $(2, 0), (3, 0), (2, \ln 2), (3, \ln 2)$. Round your answer to 4 decimal places.

Solution: We can write the rectangle as $R = [2, 3] \times [0, \ln 2]$. Hence,

$$\text{Area of } R = (3-2)(\ln 2 - 0) = \ln 2.$$

Next, we integrate. Our integral is

$$\int_2^3\int_0^{\ln 2}\frac{15\ln 2}{4}e^y\sqrt{x+e^y}\,dy\,dx$$

for which we use u-substitution.

Let $u = x + e^y$, then $du = e^y\,dy^{\diamond}$, $u(0) = x + e^0 = x + 1$, and $u(\ln 2) = x + e^{\ln 2} = x + 2$. So we write

$$\int_2^3\int_0^{\ln 2}\frac{15\ln 2}{4}e^y\sqrt{x+e^y}\,dy\,dx = \int_2^3\int_{x+1}^{x+2}\frac{15\ln 2}{4}\sqrt{u}\,du\,dx$$

$\diamond$ Because we are integrating with respect to y, we treat x as a constant — even when we use u-substitution.

$$= \int_2^3 \frac{15\ln 2}{4}\left(\frac{2}{3}u^{3/2}\Big|_{u=x+1}^{u=x+2}\right)dx$$
$$= \int_2^3 \frac{5\ln 2}{2}\left[(x+2)^{3/2}-(x+1)^{3/2}\right]dx$$
$$= \frac{5\ln 2}{2}\left[\frac{2}{5}(x+2)^{5/2}-\frac{2}{5}(x+1)^{5/2}\right]_{x=2}^{x=3}$$
$$= \ln 2\left[(3+2)^{5/2}-(3+1)^{5/2}-\left((2+2)^{5/2}-(2+1)^{5/2}\right)\right]$$
$$= \ln 2\left[5^{5/2}-2(4)^{5/2}+3^{5/2}\right]$$

Hence, the average value of $f(x,y)$ over the rectangle R is

$$\text{Ave}_f = \frac{1}{\ln 2}\int_2^3\int_0^{\ln 2}\frac{15\ln 2}{4}e^y\sqrt{x+e^y}\,dy\,dx$$
$$= \frac{1}{\ln 2}(\ln 2)\left[5^{5/2}-2(4)^{5/2}+3^{5/2}\right]$$
$$= -64+9\sqrt{3}+25\sqrt{5}\approx \boxed{7.4902}.$$

4. Suppose the function

$$P(x,t)=\frac{10{,}000e^{t/2}}{1+x}$$

describes the population of a city where x is the number of miles from the center of the city and t is the number of years after the year 2000. Find the average population of the city over the first 10 years within a radius of 5 miles from the city center. Round your answer to the nearest integer.

Solution: Here, the rectangle is

$$R = \underbrace{[0,5]}_{x}\times\underbrace{[0,10]}_{t}$$

which means the area of $R=(10-0)(5-0)=50$. We write

$$\int_0^{10}\int_0^5\frac{10{,}000e^{t/2}}{1+x}\,dx\,dt = \int_0^{10}10{,}000e^{t/2}\ln(1+x)\Big|_{x=0}^{x=5}dt$$
$$= \int_0^{10}\left(10{,}000e^{t/2}\ln(1+5)-10{,}000e^{t/2}\underbrace{\ln(1)}_{0}\right)dt$$
$$= 10{,}000\int_0^{10}\ln 6e^{t/2}\,dt$$
$$= 20{,}000\ln 6e^{t/2}\Big|_{t=0}^{t=10}$$

$$= 20{,}000 \ln 6(e^5 - e^0)$$
$$= 20{,}000 \ln 6(e^5 - 1).$$

Therefore, the average value is

$$\frac{1}{50}(20{,}000 \ln 6(e^5 - 1)) \approx \boxed{105{,}652}.$$

In general, for any region R, the average value of $f(x, y)$ over R is given by

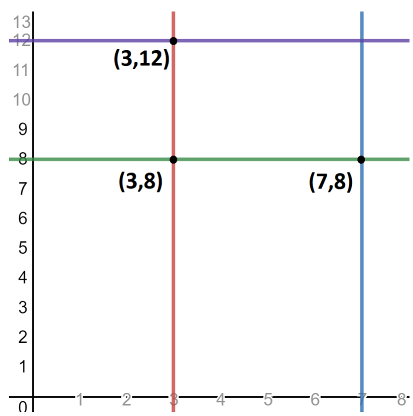
$$\text{Ave}_f = \frac{1}{\text{Area of } R} \iint_R f(x, y)\, dA.$$

3. Additional Example

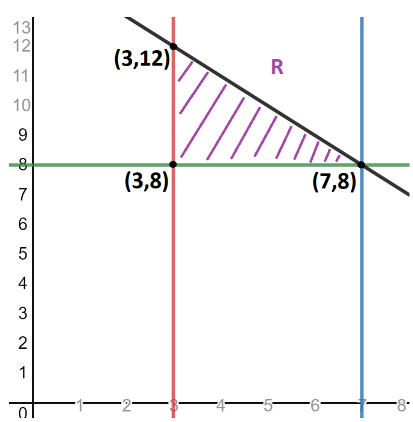
EXAMPLES.

1. Find the volume under the surface $z = xy$ above the triangle with vertices $(3, 8, 0)$, $(7, 8, 0)$, and $(3, 12, 0)$.

Solution: The vertex $(x, y, z) = (3, 8, 0)$ means $x = 3$, $y = 8$, $z = 0$. In fact, notice that for each vertex, the z-value is 0. This means that these vertices all live in the xy-plane:



The triangle R is the region marked below:



We need to find the function that describes the line between the points $(3, 12)$ and $(7, 8)$. We use the point-slope formula:

$$y - y_0 = m(x - x_0).$$

Here,

$$m = \frac{12-8}{3-7} = \frac{4}{-4} = -1.$$

Thus, we see that

$$y - 12 = -1(x-3) = -x + 3$$

implies that $y = -x + 15$.

We want to describe our region R in terms of x and y. We see that the x-values vary between 3 and 7 and that the y-values are between 8 and the line $y = -x + 15$. Therefore, we describe our region R via

$$\begin{cases} 8 \leq y \leq -x+15 \\ 3 \leq x \leq 7 \end{cases}$$

Finally, we integrate:

$$\begin{aligned}
\iint_R (xy)\,dA &= \int_3^7 \int_8^{-x+15} xy\,dy\,dx \\
&= \int_3^7 \frac{1}{2}xy^2 \Big|_{y=8}^{y=-x+15} dx \\
&= \int_3^7 \frac{1}{2}\left[x(-x+15)^2 - x(8)^2\right] dx \\
&= \int_3^7 \frac{1}{2}\left[x(x^2 - 30x + 225) - 64x\right] dx \\
&= \int_3^7 \frac{1}{2}\left[x^3 - 30x^2 + 225x - 64x\right] dx \\
&= \int_3^7 \frac{1}{2}\left[x^3 - 30x^2 + 161x\right] dx \\
&= \frac{1}{2}\left[\frac{1}{4}x^4 - \frac{30}{3}x^3 + \frac{161}{2}x^2\right]_{x=3}^{x=7} \\
&= \frac{1}{2}\left[\frac{1}{4}x^4 - 10x^3 + \frac{161}{2}x^2\right]_{x=3}^{x=7}
\end{aligned}$$

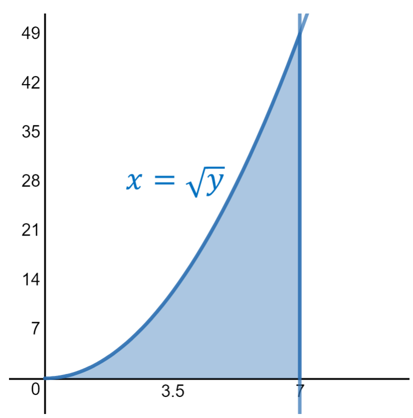
$$= \frac{1}{2}\left[\frac{1}{4}(7)^4 - 10(7)^3 + \frac{161}{2}(7)^2 - \left(\frac{1}{4}(3)^4 - 10(3)^3 + \frac{161}{2}(3)^2\right)\right]$$
$$= \frac{1}{2}(640)$$
$$= \boxed{320}$$

2. Evaluate $\displaystyle\int_0^{49}\int_{\sqrt{y}}^{7} 5\sqrt{x^3+1}\,dx\,dy$.

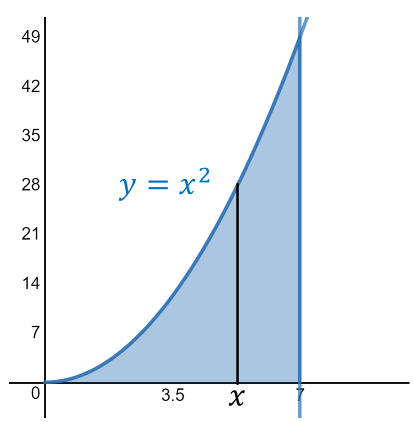
Solution: We cannot integrate as the integral is written which means we need to swap the order of integration. Our region is given as

$$\begin{cases} \sqrt{y} \leq x \leq 7 \\ 0 \leq y \leq 49 \end{cases}$$

where the dependent variable is x and the independent variable is y. We need to rewrite this where y is the dependent variable and x is the independent variable. To do this, we sketch a quick picture:



From this, it should be clear that $0 \leq x \leq 7$. As for y, we observe that if $x = \sqrt{y}$, then $y = x^2$. Our picture becomes



The black line means that for each x value between 0 and 7, the y value cannot be any larger than the curve $y = x^2$. But this means that

$$\begin{cases} 0 \leq y \leq x^2 \\ 0 \leq x \leq 7 \end{cases}$$

Now, we can write

$$\begin{aligned} \int_0^{49} \int_{\sqrt{y}}^{7} 5\sqrt{x^3+1}\, dx dy &= \int_0^7 \int_0^{x^2} 5\sqrt{x^3+1}\, dy\, dx \\ &= \int_0^7 5y\sqrt{x^3+1}\Big|_{y=0}^{y=x^2} dx \\ &= \int_0^7 5(x^2)\sqrt{x^3+1}\, dx \end{aligned}$$

This is a u-substitution problem. Take $u = x^3+1$, then $du = 3x^2\, dx \Rightarrow dx = \dfrac{du}{3x^2}$. Write

$$\begin{aligned} \int_0^7 5x^2\sqrt{x^3+1}\, dx &= \int_{u(0)}^{u(7)} 5x^2\sqrt{u}\, \underbrace{\frac{du}{3x^2}}_{dx} \\ &= \int_{u(0)}^{u(7)} \frac{5}{3}\sqrt{u}\, du \\ &= \int_{u(0)}^{u(7)} \frac{5}{3}u^{1/2}\, du \\ &= \left(\frac{5}{3}\right)\left(\frac{1}{1/2+1}\right) u^{1/2+1}\Big|_{u(0)}^{u(7)} \\ &= \frac{5}{3}\left(\frac{1}{3/2}\right) u^{3/2}\Big|_{u(0)}^{u(7)} \\ &= \frac{5}{3}\left(\frac{2}{3}\right) u^{3/2}\Big|_{u(0)}^{u(7)} \\ &= \frac{10}{9}(x^3+1)^{3/2}\Big|_0^7 \\ &= \frac{10}{9}\left[((7)^3+1)^{3/2} - ((0)+1)^{3/2}\right] \\ &= \boxed{\frac{10}{9}\left[(344)^{3/2} - 1\right]} \end{aligned}$$

3. A large building with a rectangular base has a curved roof whose height is

$$h(x,y) = 82 - 0.05x^2 + 0.026y^2.$$

The rectangular base extends from $-50 \le x \le 50$ feet and $-100 \le y \le 100$ feet. Find the average height of the building, round your answers to the nearest 3 decimal places.

Solution: The area of the rectangle over which we are integrating is

$$[50 - (-50)][100 - (-100)] = [100][200] = 20{,}000.$$

Thus, the average height of the building is

$$\frac{1}{20{,}000}\int_{-50}^{50}\int_{-100}^{100}(82 - 0.05x^2 + 0.026y^2)\,dy\,dx$$

$$= \frac{1}{20{,}000}\int_{-50}^{50}\left[82y - 0.05x^2y + \frac{0.026}{3}y^3\right]_{y=-100}^{y=100} dx$$

$$= \frac{1}{20{,}000}\int_{-50}^{50}\Bigg[82(100) - 0.05x^2(100) + \frac{0.026}{3}(100)^3$$

$$- \left(82(-100) - 0.05x^2(-100) + \frac{0.026}{3}(-100)^3\right)\Bigg] dx$$

$$= \frac{1}{20{,}000}\int_{-50}^{50}\left[82(100) - 5x^2 + \frac{26{,}000}{3} - \left(-82(100) + 5x^2 - \frac{26{,}000}{3}\right)\right] dx$$

$$= \frac{1}{20{,}000}\int_{-50}^{50}\left[16{,}400 + \frac{52{,}000}{3} - 10x^2\right] dx$$

$$= \frac{1}{20{,}000}\left[16{,}400x + \frac{52{,}000}{3}x - \frac{10}{3}x^3\right]_{x=-50}^{x=50}$$

$$= \frac{1}{20{,}000}\left[16{,}400(50) + \frac{52{,}000}{3}(50) - \frac{10}{3}(50)^3 - \left(16{,}400(-50) + \frac{52{,}000}{3}(-50) - \frac{10}{3}(-50)^3\right)\right]$$

$$= \frac{1}{20{,}000}\left[820{,}000 + \frac{2{,}600{,}000}{3} - \frac{1{,}250{,}000}{3} - \left(-820{,}000 - \frac{2{,}600{,}000}{3} + \frac{1{,}250{,}000}{3}\right)\right]$$

$$= \frac{1}{20{,}000}\left[1{,}640{,}000 + \frac{2{,}700{,}000}{3}\right]$$

$$= \boxed{127 \text{ feet}}$$

Lesson 30: Systems of Equations, Matrices, Gaussian Elimination

1. Solutions to In-Class Examples

A system of equations is just a list of equations. The goal is to find the inputs which make the list true: we call these inputs **solutions**.

Types of Solutions

Inconsistent	**Consistent Independent**	**Consistent Dependent**
There are no solutions	There is 1 solution	There are many solutions
Ex $\begin{cases} x+y = 1 \\ x+y = -1 \end{cases}$	**Ex** $\begin{cases} x+y = 2 \\ -x+y = 0 \end{cases}$	**Ex** $\begin{cases} x+y = 1 \\ 2x+2y = 2 \end{cases}$
No (x, y) work	$(x, y) = (1, 1)$	$(x, y) = (1-t, t)$ for any t

The following 3 operations on a system of equations will not change the set of solutions:

(1) switching the order of the equations

(2) multiplying an equation by a **non-zero** constant

(3) adding a multiple of one equation to another equation

The method of solving a system of equations by algebraic manipulation is called the **elimination method**.

We can write a system of equations as an **augmented matrix**.

A matrix looks like $\begin{bmatrix} 1 & 0 \\ 3 & 12 \end{bmatrix}$ and an augmented matrix looks like $\left[\begin{array}{cc|c} 6 & -1 & 3 \\ 2 & 0 & -7 \end{array}\right]$

Ex 1.

$$\begin{array}{ccc} x & y & \text{const} \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 3 & 12 & 9 \end{array}\right]$$
$$\updownarrow$$
$$\begin{array}{rcrcr} 1x & + & 0y & = & 7 \\ 3x & + & 12y & = & 9 \end{array}$$

$$\begin{array}{cccc} x & y & z & \text{const} \end{array}$$
$$\left[\begin{array}{ccc|c} 6 & -1 & 3 & 0 \\ 2 & 0 & -7 & 6 \end{array}\right]$$
$$\updownarrow$$
$$\begin{array}{rcrcrcr} 6x & + & (-1)y & + & 3z & = & 0 \\ 2x & + & (0)y & + & (-7)z & = & 6 \end{array}$$

We can do similar operations to matrices as we can to systems of equations. We call them **row operations**. We may

(1) switch two rows ($R_1 \leftrightarrow R_2$)

(2) multiply a row by a **non-zero** constant ($2R_1 \to R_1$)

(3) add a multiple of one row to another row ($3R_1 + R_2 \to R_2$)

We can also solve systems of equations when they are in matrix form. To solve these systems of equations using matrices, we put matrices into **row-echelon form**:

Consistent Independent	**Consistent Dependent**	**Inconsistent**
$\left[\begin{array}{cc\|c} 1 & \# & \# \\ 0 & 1 & \# \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & \# & \# \\ 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & \# & \# \\ 0 & 0 & \diamond \end{array}\right]$
$\left[\begin{array}{ccc\|c} 1 & \# & \# & \# \\ 0 & 1 & \# & \# \\ 0 & 0 & 1 & \# \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & \# & \# & \# \\ 0 & 1 & \# & \# \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & \# & \# & \# \\ 0 & 1 & \# & \# \\ 0 & 0 & 0 & \diamond \end{array}\right]$

where # is any number and $\diamond$ is any non-zero number.

Solving a system of equations by putting a matrix in row-echelon form is called **Gaussian elimination**.

EXAMPLES.

1. Solve the following system of equations using matrices:

$$\begin{cases} 3x + 2y = 7 \\ 6x + 3y = 12 \end{cases}$$

$$\xrightarrow{\text{Translate}} \left[\begin{array}{cc|c} 3 & 2 & 7 \\ 6 & 3 & 12 \end{array}\right] \xrightarrow{-2R_1+R_2\to R_2} \left[\begin{array}{cc|c} 3 & 2 & 7 \\ 0 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{-R_2\to R_2} \left[\begin{array}{cc|c} 3 & 2 & 7 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{\frac{1}{3}R_1\to R_1} \left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{7}{3} \\ 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\text{Translate}} \begin{cases} x + \frac{2}{3}y = \frac{7}{3} \\ y = 2 \end{cases}$$

Since $y = 2$, we can substitute into the first equation to get

$$x + \frac{2}{3}(2) = \frac{7}{3} \quad \Rightarrow \quad x = \frac{7}{3} - \frac{4}{3} = \frac{3}{3} = 1.$$

Solution: $(x, y) = (1, 2)$

2. Solve

$$\begin{cases} 2x + 6y = 10 \\ 3x + 5y = 11 \end{cases}$$

$$\xrightarrow{\text{Translate}} \left[\begin{array}{cc|c} 2 & 6 & 10 \\ 3 & 5 & 11 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 3 & 5 & 11 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_2\to R_2} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -4 & -4 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\text{Translate}} \begin{cases} x + 3y = 5 \\ y = 1 \end{cases}$$

Substituting $y = 1$ into the first equation, we get

$$x + 3(1) = 5 \quad \Rightarrow \quad x = 5 - 3 = 2.$$

Solution: $(x, y) = (2, 1)$

3. Put the following matrix into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 0 & 5 \\ 2 & 5 & 5 & 29 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 0 & 5 \\ 0 & 9 & -1 & 11 \end{array}\right] \xrightarrow{-9R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & -1 & -34 \end{array}\right]$$

$$\xrightarrow{-R_3\to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 34 \end{array}\right]$$

4. A goldsmith has two alloys of gold with the first having a purity of 90% and the second having a purity of 70%. If x grams of the first are mixed with y grams of the second such that we get 100 grams of an alloy containing 80% gold, find x to the nearest gram.

Solution: This question comes down to correctly setting up the system of equations which describes this situation. We want, in total, 100 grams of the alloy. Hence, $x + y = 100$. Out of these 100 grams, we want 80% to be gold. But each gram of the first alloy only contributes .9 grams of gold and each gram of the second alloy only contributes .7 grams of gold. Therefore, the second equation in our system is $.9x + .7y = 80$. Our system is then

$$\begin{cases} x + y &= 100 \\ .9x + .7y &= 80 \end{cases}$$

Instead of matrices, we use the elimination method. The first equation tells us that $x = 100 - y$ so by the second equation

$$\begin{aligned} .9(100 - y) + .7y &= 80 \\ \Rightarrow \quad 90 - .9y + .7y &= 80 \\ \Rightarrow \quad 90 - .2y &= 80 \\ \Rightarrow \quad -.2y &= -10 \\ \Rightarrow \quad y &= 50 \end{aligned}$$

Thus, since $x = 100 - y$, if $y = 50$ we know that $\boxed{x = 50}$.

5. Solve and classify the following system of equations:

$$\begin{cases} 3x + 2y + z = 1 \\ x + y + 2z = 0 \\ 4x + 3y + 3z = 1 \end{cases}$$

$$\xrightarrow{\text{Translate}} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 4 & 3 & 3 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 1 \\ 4 & 3 & 3 & 1 \end{array}\right] \xrightarrow{-4R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 1 \\ 0 & -1 & -5 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If we translate this back into a system of equations, we get

$$\begin{cases} x + y + 2z = 0 \\ y + 5z = -1 \\ 0x + 0y + 0z = 0 \end{cases}$$

We see that any (x, y, z) will satisfy the last equation. This is an example of a **consistent dependent** system, which means we have infinitely many solutions. In this situation, take $z = t$ (here, t is called the **free parameter**). The first two equations become

$$\begin{cases} x + y + 2t = 0 \\ y + 5t = -1 \end{cases}$$

The second equation implies that $y = -1 - 5t$. Substituting this into the first equation, we get

$$\begin{aligned} x + (-1 - 5t) + 2t &= 0 \\ \Rightarrow \quad x - 1 - 3t &= 0 \\ \Rightarrow \quad x &= 1 + 3t \end{aligned}$$

Solution: $x = 1 + 3t, \quad y = -1 - 5t, \quad z = t$

2. Additional Examples

EXAMPLES.

1. Four sandwiches and two bags of chips contain 848 calories. One sandwich and one bag of chips contain 305 calories. How many calories are there in a sandwich? Solve using Gaussian elimination.

   Solution: Let x be the number of calories in one sandwich and let y be the number calories in one bag of chips. Our goal is to find x. The information above can be translated into the following system of equations:

$$\begin{cases} x + y &= 305 \\ 4x + 2y &= 848 \end{cases}$$

   Translating this into a matrix, we put it in row-echelon form:

$$\left[\begin{array}{cc|c} 1 & 1 & 305 \\ 4 & 2 & 848 \end{array}\right] \xrightarrow{-4R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 305 \\ 0 & -2 & -372 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 305 \\ 0 & 1 & 186 \end{array}\right]$$

   Translating this back into a system of equations, we have

$$\begin{cases} x + y &= 305 \\ y &= 186 \end{cases}$$

   By the first equation $x = 305 - y$, which means that

$$x = 305 - 186 = \boxed{119 \text{ calories}}$$

2. Solve the following problem using Gaussian elimination: An object is moving vertically where a is the constant acceleration, and for $t = 0$ that v is the initial velocity and h is the initial height. Given that at $t = 1$ second, $s = 47$ feet; at $t = 2$ seconds, $s = 85$ feet; and at $t = 3$ seconds, $s = 47$ feet. Find a function for the height, s, that is modeled using $s(t) = \frac{1}{2}at^2 + vt + h$.

   Solution: We are given the function $s(t) = \frac{1}{2}at^2 + vt + h$ where we want to solve for a, v, h. We also know the $s(t)$ values for $t = 1, 2, 3$. So we have the system of equations

$$\begin{cases} \frac{1}{2}(1)^2 a & + & (1)v & + & h & = & 47 \text{ where } t = 1 \\ \frac{1}{2}(2)^2 a & + & (2)v & + & h & = & 85 \text{ where } t = 2 \\ \frac{1}{2}(3)^2 a & + & (3)v & + & h & = & 47 \text{ where } t = 3 \end{cases}$$

In a matrix, this becomes

$$\begin{array}{cccc} a & v & h & s(t) \end{array}$$
$$\left[\begin{array}{ccc|c} \frac{1}{2}(1)^2 & 1 & 1 & 47 \\ \frac{1}{2}(2)^2 & 2 & 1 & 85 \\ \frac{1}{2}(3)^2 & 3 & 1 & 47 \end{array}\right] = \left[\begin{array}{ccc|c} \frac{1}{2} & 1 & 1 & 47 \\ 2 & 2 & 1 & 85 \\ \frac{9}{2} & 3 & 1 & 47 \end{array}\right]$$

We put this matrix into row-echelon form.

$$\xrightarrow[2R_3 \to R_3]{2R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 94 \\ 2 & 2 & 1 & 85 \\ 9 & 6 & 2 & 94 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 94 \\ 0 & -2 & -3 & -103 \\ 9 & 6 & 2 & 94 \end{array}\right]$$

$$\xrightarrow{-9R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 94 \\ 0 & -2 & -3 & -103 \\ 0 & -12 & -16 & -752 \end{array}\right] \xrightarrow{-6R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 94 \\ 0 & -2 & -3 & -103 \\ 0 & 0 & 2 & -134 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 94 \\ 0 & -2 & -3 & -103 \\ 0 & 0 & 1 & -67 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 94 \\ 0 & 1 & \frac{3}{2} & \frac{103}{2} \\ 0 & 0 & 1 & -67 \end{array}\right]$$

Translating back to a system of equations, we have

$$\begin{cases} a & + & 2v & + & 2h & = & 94 \\ & & v & + & \frac{3}{2}h & = & \frac{103}{2} \\ & & & & h & = & -67 \end{cases}$$

Since $h = -67$, substituting into the second equation gives us

$$v + \frac{3}{2}(-67) = \frac{103}{2} \quad \Rightarrow \quad v = \frac{103}{2} + \frac{201}{2} = \frac{304}{2} = 152.$$

Hence, by the first equation,

$$a + 2(152) + 2(-67) = 94 \quad \Rightarrow \quad a = 94 - 304 + 134 = -76.$$

Therefore, our function $s(t)$ is given by

$$s(t) = \frac{1}{2}(-76)t^2 + 152t - 67 = \boxed{-38t^2 + 152t - 67}.$$

Lesson 31: Gauss-Jordan Elimination

1. Solutions to In-Class Examples

A matrix is in **reduced row-echelon form** if it looks like one of the following:

Consistent Independent	Consistent Dependent	Inconsistent
$\left[\begin{array}{cc\|c} 1 & 0 & \# \\ 0 & 1 & \# \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & 0 & \# \\ 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & 0 & \# \\ 0 & 0 & \diamond \end{array}\right]$
$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 0 & \diamond \end{array}\right]$

where $\#$ is any number and $\diamond$ is any non-zero number. The method of solving a system of equations by putting its augmented matrix into reduced row-echelon form is called **Gauss-Jordan elimination**.

EXAMPLES.

1. Use Gauss-Jordan elimination to solve $\begin{cases} 2x + 3y &= -5 \\ -x + 2y &= -8 \end{cases}$

$$\xrightarrow{\text{Translate}} \left[\begin{array}{cc|c} 2 & 3 & -5 \\ -1 & 2 & -8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} -1 & 2 & -8 \\ 2 & 3 & -5 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 2 & 3 & -5 \end{array}\right]$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 7 & -21 \end{array}\right] \xrightarrow{\frac{1}{7}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 1 & -3 \end{array}\right] \xrightarrow{2R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

Solution: $(x, y) = (2, -3)$

2. Put the following matrix into reduced row-echelon form: $\left[\begin{array}{ccc|c} -2 & 3 & 3 & -4 \\ 1 & -1 & 2 & 5 \\ -1 & 2 & -1 & -5 \end{array}\right]$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ -2 & 3 & 3 & -4 \\ -1 & 2 & -1 & -5 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 1 & 7 & 6 \\ -1 & 2 & -1 & -5 \end{array}\right]$$

$$\xrightarrow{R_1+R_3\to R_3} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 5 \\ 0 & 1 & 7 & 6 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-R_2+R_3\to R_3} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 5 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & -6 & -6 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{6}R_3\to R_3} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 5 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-7R_3+R_2\to R_2} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_3+R_1\to R_1} \left[\begin{array}{rrr|r} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2+R_1\to R_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

3. Solve the following using Gauss-Jordan elimination:

$$\begin{cases} 3x - 2y - 6z = 1 \\ x + 2y + z = 0 \\ -x + 2y - z = 4 \end{cases}$$

$$\xrightarrow{\text{Translate}} \left[\begin{array}{rrr|r} 3 & -2 & -6 & 1 \\ 1 & 2 & 1 & 0 \\ -1 & 2 & -1 & 4 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 3 & -2 & -6 & 1 \\ -1 & 2 & -1 & 4 \end{array}\right]$$

$$\xrightarrow{-3R_1+R_2\to R_2} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -8 & -9 & 1 \\ -1 & 2 & -1 & 4 \end{array}\right] \xrightarrow{R_1+R_3\to R_3} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -8 & -9 & 1 \\ 0 & 4 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & -8 & -9 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}R_2\to R_2} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -8 & -9 & 1 \end{array}\right]$$

$$\xrightarrow{8R_2+R_3\to R_3} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -9 & 9 \end{array}\right] \xrightarrow{-\frac{1}{9}R_3\to R_3} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-R_3+R_1\to R_1} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{-2R_2+R_1\to R_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Solution: $(x, y, z) = (-1, 1, -1)$

4. Use Gauss-Jordan elimination to solve the system of equations:

$$\begin{cases} x + y + z = 14 \\ 5x + 2y + 5z = 52 \\ y - 2z = 2 \end{cases}$$

$$\xrightarrow{\text{Translate}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 14 \\ 5 & 2 & 5 & 52 \\ 0 & 1 & -2 & 2 \end{array}\right] \xrightarrow{-5R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 14 \\ 0 & -3 & 0 & -18 \\ 0 & 1 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{-\frac{R_2}{3} \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 6 \\ 0 & 1 & -2 & 2 \end{array}\right] \xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-R_3+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 12 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(x, y, z) = (6, 6, 2)$

Lesson 32: Matrix Operations

1. Solutions to In-Class Examples

The dimensions of a matrix are always given by row×column.

Ex 1.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 7 & 2 \end{bmatrix}$$
2×3 matrix

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$
3×1 matrix

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
2×2 matrix

Elements in a matrix are specified by the ordered pair (row, column).

Ex 2. 6 is the $(2, 3)$-entry of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Matrix Addition: We add two matrices component-wise, that is, by adding each entry that has the same (row, column). We can only add matrices that have the same dimensions.

Ex 3.

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 2+(3) & 1+(-1) \\ -1+(0) & 3+(-5) \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -5 \\ 2 & 3 & 4 \\ -1 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -1+1 & 3+1 & 2+(-5) \\ 0+2 & 1+3 & -1+4 \\ 1+(-1) & 2+(-2) & 5+(-1) \end{bmatrix} = \begin{bmatrix} 0 & 4 & -3 \\ 2 & 4 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

Scalar Multiplication: A **scalar** is a number that isn't in a matrix. We use the term scalar to differentiate it from the entries of a matrix. We can multiply matrices by scalars, which amounts to multiplying each entry in the matrix by the scalar.

Ex 4.

$$3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) \\ 3(-1) & 3(3) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -3 & 9 \end{bmatrix}$$

$$2\begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & 0 \\ 0 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(1) & 2(-1) \\ 2(-1) & 2(3) & 2(0) \\ 2(0) & 2(7) & 2(5) \end{bmatrix} = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 6 & 0 \\ 0 & 14 & 10 \end{bmatrix}$$

Matrix Multiplication: We can multiply matrices together. This is **not** done component-wise. There is an excellent reason why we do matrix multiplication this way but the reason is beyond the scope of this class.

Ex 5. If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix},$$

find AB.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(3) + 0(-1) + 0(2) \\ 2(3) + 1(-1) + 3(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

Notice that in terms of the dimensions of the matrix, we have $(2\times3)(3\times1) = 2\times1$. This is true in general. In a similar way, $(5 \times 2)(2 \times 3) = 5 \times 3$.

Sometimes matrix multiplication doesn't make sense. For example, BA doesn't make sense because the number of **columns on the left** has to equal the number of **rows on the right**.

EXAMPLES.

1. Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 4 & -1 \end{bmatrix}$. Find $3A$, $3A - B$, AB, and BA.

$$3A = 3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3(2) & 3(1) \\ 3(-1) & 3(0) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 0 \end{bmatrix}$$

$$3A - B = 3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-3 & 3-0 \\ -3-4 & 0-(-1) \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -7 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 2(3)+1(4) & 2(0)+1(-1) \\ -1(3)+0(4) & -1(0)+0(-1) \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ -3 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 0 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3(2)+0(-1) & 3(1)+0(0) \\ 4(2)+(-1)(-1) & 4(1)+(-1)(0) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 9 & 4 \end{bmatrix}$$

In general, $AB \neq BA$. So order matters for matrix multiplication.

2. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$, find A^2. What is the $(3,2)$-entry of A^2?

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+0(1)+1(-1) & 1(0)+0(2)+1(-1) & 1(1)+0(-1)+1(3) \\ 1(1)+2(1)+(-1)(-1) & 1(0)+2(2)+(-1)(-1) & 1(1)+2(-1)+(-1)(3) \\ -1(1)+(-1)(1)+3(-1) & -1(0)+(-1)(2)+3(-1) & -1(1)+(-1)(-1)+3(3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 4 \\ 4 & 5 & -4 \\ -5 & -5 & 9 \end{bmatrix}$$

The $(3,2)$-entry is $\boxed{-5}$.

3. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$, find AB.

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)+(-1)(-1)+2(2) & 1(1)+(-1)(0)+2(3) \\ 0(1)+3(-1)+(-2)(2) & 0(1)+3(0)+(-2)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 7 \\ -7 & -6 \end{bmatrix}$$

4. Let $M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Find $M^2 - 3M$.

$$\begin{aligned} M^2 - 3M &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+(-1)(1) & 1(-1)+(-1)(0) \\ 1(1)+1(0) & 1(-1)+0(0) \end{bmatrix} - \begin{bmatrix} 3(1) & 3(-1) \\ 3(1) & 3(0) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ -2 & -1 \end{bmatrix} \end{aligned}$$

2. Additional Examples

EXAMPLES.

1. Given the number of calories expended by people with different weights and using different exercises for 20 minute time periods is

$$A = \begin{array}{c} \begin{array}{cc} 120\text{lb} & 150\text{lb} \end{array} \\ \begin{bmatrix} 114 & 124 \\ 116 & 158 \\ 78 & 74 \end{bmatrix} \end{array} \begin{array}{l} \text{Bicycling} \\ \text{Jogging} \\ \text{Walking} \end{array}$$

A 120-pound person and a 150-pound person both bicycle for 40 minutes, jog for 10 minutes, and walk for 60 minutes. Create a matrix for the time spent exercising, then multiply the matrices to find the number of calories expended by the 120-pound and the 150-pound person. Round your answers to 2 decimal places.

Solution: Since we are measuring time in 20 minute periods, the matrix describing the time spent exercising per exercise is

$$\begin{array}{ccc} \text{bicycle} & \text{jog} & \text{walk} \\ [2 & .5 & 3] \end{array}$$

Then we see that

$$\begin{aligned} \begin{bmatrix} 2 & .5 & 3 \end{bmatrix}\begin{bmatrix} 114 & 124 \\ 116 & 158 \\ 78 & 74 \end{bmatrix} &= \begin{bmatrix} 114(2)+116(.5)+78(3) \\ 124(2)+158(.5)+74(3) \end{bmatrix} \begin{array}{l} 120\text{lb} \\ 150\text{lb} \end{array} \\ &= \begin{bmatrix} 520 \\ 549 \end{bmatrix} \begin{array}{l} 120\text{lb} \\ 150\text{lb} \end{array} \end{aligned}$$

Thus, we see that the 120 pound person expends 520 calories and the 150 pound person expends 549 calories.

Lesson 33: Inverses and Determinants of Matrices (I)

1. Solutions to In-Class Examples

DEFINITION 69. The square matrix with 1s along the diagonal and 0s elsewhere is called the **identity matrix**.

Ex 1.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
2 × 2 identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
3 × 3 identity matrix

If I_n is the $n \times n$ identity matrix, then for any $n \times n$ matrix A,

$$AI_n = A = I_n A.$$

For some square matrices A, there exists a **inverse matrix** A^{-1}, i.e.,

$$AA^{-1} = I_n = A^{-1}A.$$

Method for Finding Matrix Inverses

Let A be an $n \times n$ matrix. Create a new matrix

$$B = \left[\begin{array}{c|c} A & I_n \end{array}\right].$$

Use row-operations to put B into **reduced-row echelon form**. If A has an inverse, A^{-1}, then the resulting matrix, B', will be of the form

$$B' = \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right].$$

Ex 2. Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$. Find A^{-1}.

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.$$

Quick Check: Show that

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A^{-1}A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

EXAMPLES.

1. Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & 0 & -1 \end{bmatrix}$, find A^{-1} if it exists.

Solution: From our method above, our matrix B is $\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$.

We put this in reduced row-echelon form.

$$\xrightarrow{R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -5 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \xrightarrow{-3R_3+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{5} & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

$$\xrightarrow{R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{3}{5} & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & -\frac{1}{5} & -\frac{4}{5} \\ 0 & 1 & 0 & \frac{3}{5} & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} & -\frac{4}{5} \\ \frac{3}{5} & \frac{2}{5} & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}.$$

Finding Solutions using Inverse Matrices

A column vector is a matrix of the form

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 8 \\ 2 \end{bmatrix}.$$

Let $AX = Y$ be a system of equations where A is the $n \times n$ coefficient matrix and X and Y are column vectors, then

$$X = A^{-1}Y.$$

EXAMPLES.

2. Find the solution of $\begin{cases} 2x + 3y &= -5 \\ -x + 2y &= -8 \end{cases}$ using inverse matrices.

Solution: Here $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $Y = \begin{bmatrix} -5 \\ -8 \end{bmatrix}$. By the method described above, $X = A^{-1}Y$. So we compute A^{-1}.

$$\left[\begin{array}{rr|rr} 2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rr|rr} -1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{rr|rr} 1 & -2 & 0 & -1 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{rr|rr} 1 & -2 & 0 & -1 \\ 0 & 7 & 1 & 2 \end{array}\right] \xrightarrow{\frac{1}{7}R_2 \to R_2} \left[\begin{array}{rr|rr} 1 & -2 & 0 & -1 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} \end{array}\right]$$

$$\xrightarrow{2R_2 + R_1 \to R_1} \left[\begin{array}{rr|rr} 1 & 0 & \frac{2}{7} & -\frac{3}{7} \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$. Then, since $X = A^{-1}Y$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} -5 \\ -8 \end{bmatrix} = \begin{bmatrix} \frac{2}{7}(-5) + \left(-\frac{3}{7}\right)(-8) \\ \frac{1}{7}(-5) + \left(\frac{2}{7}\right)(-8) \end{bmatrix} = \begin{bmatrix} \frac{14}{7} \\ -\frac{21}{7} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Solution: $(x, y) = (2, -3)$

3. Find a solution to $\begin{cases} -x+5y+2z &= 39 \\ 3y+5z &= 39 \\ 2x+y+2z &= 28 \end{cases}$ given that the inverse of the coefficient matrix is

$$\frac{1}{37}\begin{bmatrix} 1 & -8 & 19 \\ 10 & -6 & 5 \\ -6 & 11 & -3 \end{bmatrix}.$$

Solution: Here,

$$A = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 3 & 5 \\ 2 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad Y = \begin{bmatrix} 39 \\ 39 \\ 28 \end{bmatrix}$$

We are also told that

$$A^{-1} = \frac{1}{37}\begin{bmatrix} 1 & -8 & 19 \\ 10 & -6 & 5 \\ -6 & 11 & -3 \end{bmatrix}$$

By our method we know that $X = A^{-1}Y$, so we can write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 1 & -8 & 19 \\ 10 & -6 & 5 \\ -6 & 11 & -3 \end{bmatrix}\begin{bmatrix} 39 \\ 39 \\ 28 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 1(39)+(-8)(39)+19(28) \\ 10(39)+(-6)(39)+5(28) \\ -6(39)+11(39)+(-3)(28) \end{bmatrix}$$

$$= \frac{1}{37}\begin{bmatrix} 259 \\ 296 \\ 111 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix}$$

Solution: $(x, y, z) = (7, 8, 3)$

Lesson 34: Inverses and Determinants of Matrices (II)

1. Solutions to In-Class Examples

The **determinant** of a matrix A ($\det A$ or $|A|$) is a function on square matrices that returns a number, **not** a matrix.

FACT 70.

- If $\det A \neq 0$, then A^{-1} exists.
- If A^{-1} exists, then $\det A \neq 0$.

DEFINITION 71. A matrix is called **singular** if $\det A = 0$. A matrix is called **non-singular** if $\det A \neq 0$.

Determinant of 2×2 Matrices:

Important 2×2 Formulas

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

(i) $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(ii) $A^{-1} = \dfrac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $\det A \neq 0$

EX 1. Let $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$. Find $\det A$ and, if it exists, find A^{-1}.

We write

$$\det A = |A| = \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = (2)(1) - (0)(-1) = 2.$$

Because $\det A \neq 0$, we know that A^{-1} exists. Thus, by (ii),

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Determinant of 3×3 Matrices: The determinant of 3×3 matrices is defined using 2×2 matrices. We compute the **minors** and **cofactors** of the matrix. Let

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & -2 & -2 \\ 1 & 0 & 1 \end{bmatrix}.$$

The **minor** of the $(3,2)$-entry, M_{32}, is the determinant of the matrix A after deleting the 3rd row and 2nd column, that is,

$$\underset{(3,2)\text{-entry}}{\begin{bmatrix} 0 & 2 & 1 \\ 2 & -2 & -2 \\ 1 & \boxed{0} & 1 \end{bmatrix}} \longrightarrow \underset{\substack{3^{\text{rd}}\text{ Row,} \\ 2^{\text{nd}}\text{ Column}}}{\begin{bmatrix} 0 & 2 & 1 \\ 2 & -2 & -2 \\ 1 & \boxed{0} & 1 \end{bmatrix}} \longrightarrow \underset{M_{32}}{\begin{vmatrix} 0 & 1 \\ 2 & -2 \end{vmatrix}} = (0)(-2)-(1)(2) = -2 = M_{32}$$

The **cofactor** of the $(3,2)$-entry, C_{32}, is $(-1)^{3+2}M_{23} = (-1)^5(-2) = 2 = C_{32}$

Ex 2. Consider the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$.

Minors and Cofactors of A for Row 2

$(2,1)$-entry

$$\begin{bmatrix} 1 & 2 & -2 \\ \boxed{0} & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$$

$$\begin{aligned} M_{21} &= \begin{vmatrix} 2 & -2 \\ 3 & 2 \end{vmatrix} \\ &= 4-(-6) = 10 \\ C_{21} &= (-1)^{2+1}M_{21} \\ &= -10 \end{aligned}$$

$(2,2)$-entry

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & \boxed{1} & 0 \\ -1 & 3 & 2 \end{bmatrix}$$

$$\begin{aligned} M_{22} &= \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} \\ &= 2-(2) = 0 \\ C_{22} &= (-1)^{2+2}M_{22} \\ &= 0 \end{aligned}$$

$(2,3)$-entry

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & \boxed{0} \\ -1 & 3 & 2 \end{bmatrix}$$

$$\begin{aligned} M_{23} &= \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \\ &= 3-(-2) = 5 \\ C_{23} &= (-1)^{2+3}M_{23} \\ &= -5 \end{aligned}$$

$$\det A = [(2,1)\text{-entry}]C_{21} + [(2,2)\text{-entry}]C_{22} + [(2,3)\text{-entry}]C_{23} = 0(-10)+1(0)+0(-5) = 0$$

Determinant of a 3×3 Matrix:

If A is a 3×3 matrix,then for any row r,

$$\det A = [(r,1)\text{-entry}]C_{r1} + [(r,2)\text{-entry}]C_{r2} + [(r,3)\text{-entry}]C_{r3}.$$

EXAMPLES.

1. Is $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ singular?

Solution: A matrix is singular if $\det A = 0$. So, we write

$$\det A = \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = (2)(-1)-(1)(0) = -2 \neq 0.$$

Therefore, A is non-singular.

2. Find the minor and cofactor of the $(1,3)$-entry of $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix} \longrightarrow M_{13} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = 0(-1) - 2(3) = \boxed{-6}$$

$$C_{13} = (-1)^{1+3} M_{13} = \boxed{-6}$$

3. Find the determinant of A from # 2.

Solution: We expand along row 1.

$(1,1)$-entry	$(1,2)$-entry	$(1,3)$-entry
$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$
$M_{11} = \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}$	$M_{12} = \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}$	$M_{13} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix}$
$= 0 - (-1) = 1$	$= 0 - 3 = -3$	$= 0 - 6 - 6$
$C_{11} = (-1)^{1+1} M_{11}$	$C_{12} = (-1)^{1+2} M_{12}$	$C_{13} = (-1)^{1+3} M_{13}$
$= 1$	$= 3$	$= -6$

Thus,

$$\det A = 1(1) + (-1)(3) + 1(-6) = 1 - 3 - 6 = \boxed{-8}.$$

4. Given $\begin{vmatrix} x-3 & 3 \\ 0 & x+1 \end{vmatrix} = 0$, find x.

Solution: Write

$$\begin{vmatrix} x-3 & 3 \\ 0 & x+1 \end{vmatrix} = (x-3)(x+1) - 0(3) = (x-3)(x+1).$$

Hence, $(x-3)(x+1) = 0$ implies $\boxed{x = -1, 3}$.

5. Given $\begin{vmatrix} x-6 & 0 & -2 \\ 33 & x+4 & 1 \\ -3 & 2 & x-6 \end{vmatrix} = 0$, find x.

We compute C_{11}, C_{12}, and C_{13}. Write

$$\begin{aligned} C_{11} &= (-1)^{1+1}\begin{vmatrix} x+4 & 1 \\ 2 & x-6 \end{vmatrix} \\ &= (x+4)(x-6)-(1)(2) \\ &= x^2-2x-24-2 \\ &= x^2-2x-26 \end{aligned}$$

$$\begin{aligned} C_{12} &= (-1)^{1+2}\begin{vmatrix} 33 & 1 \\ -3 & x-6 \end{vmatrix} \\ &= -[(33)(x-6)-(-3)(1)] \\ &= -[33x-198+3] \\ &= -33x+195 \end{aligned}$$

$$\begin{aligned} C_{13} &= (-1)^{1+3}\begin{vmatrix} 33 & x+4 \\ -3 & 2 \end{vmatrix} \\ &= (33)(2)-(x+4)(-3) \\ &= 66+3x+12 \\ &= 3x+78 \end{aligned}$$

Therefore,

$$\begin{aligned} \begin{vmatrix} x-6 & 0 & -2 \\ 33 & x+4 & 1 \\ -3 & 2 & x-6 \end{vmatrix} &= (x-6)\underbrace{(x^2-2x-26)}_{C_{11}}+(0)\underbrace{(-33x+195)}_{C_{12}}+(-2)\underbrace{(3x+78)}_{C_{13}} \\ &= x^3-8x^2-14x+156-6x-156 \\ &= x^3-8x^2-20x \\ &= x(x^2-8x-20) \\ &= x(x+2)(x-10) \end{aligned}$$

Because $\begin{vmatrix} x-6 & 0 & -2 \\ 33 & x+4 & 1 \\ -3 & 2 & x-6 \end{vmatrix} = 0$, we have $x(x+2)(x-10)=0$ and so $\boxed{x=-2,0,10}$.

Lesson 35: Eigenvectors and Eigenvalues (I)

0. Basic Definitions

DEFINITION 72. A vector is a matrix with only one column.

Ex 1. $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are vectors.

DEFINITION 73. For A a square matrix, there exists a number λ and a collection of (non-zero) vectors $\vec{v}_\lambda$ such that

$$A\vec{v}_\lambda = \lambda\vec{v}_\lambda.$$

We say λ is an **eigenvalue** of A and $\vec{v}_\lambda$ is an **eigenvector associated to** λ.

NOTE 74. The vectors $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **do not** count as eigenvectors.

Ex 2. Let $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $\lambda = 2$, and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(1) + (-1)(1) \\ 2(1) + 0(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\lambda = 2$ is an **eigenvalue** of A and the vector $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an **eigenvector** associated to $\lambda = 2$.

Ex 3. Is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 4 & -2 \\ -21 & 3 \end{bmatrix}$ associated to the eigenvalue $r = -3$?

Solution: $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is an eigenvector of this matrix associated to $r = -3$ **if** the following equation is true:

$$\begin{bmatrix} 4 & -2 \\ -21 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \stackrel{?}{=} -3 \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

We check:

$$\begin{bmatrix} 4 & -2 \\ -21 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4(-1) + (-2)(3) \\ -21(-1) + 3(3) \end{bmatrix}$$

$$= \begin{bmatrix} -10 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Hence, we see that $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is **not** an eigenvector associated to $r = -3$. It **is** an eigenvector associated to $r = 10$, but this was not the question.

Homework Structure

There are three types of problem on this homework:

(1) match the eigenvector to this matrix

(2) find the eigenvalues of this matrix

(3) find the eigenvalues and eigenvectors of this matrix

1. Matching Eigenvectors to Matrices

EXAMPLES.

1. Which of the following are eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix}?$$

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution: What does it mean to be an eigenvector? It means there exists a number λ such that

$$\begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

$\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector **only if** there is such a λ. We check by matrix multiplication whether this is true. Write

$$\begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 5(6) + (-12)(2) \\ 2(6) + (-5)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 30-24 \\ 12-10 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 6 \\ 2 \end{bmatrix} \text{ eigenvector}$$

$$\begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5(3)+(-12)(3) \\ 2(3)+(-5)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 15-36 \\ 6-15 \end{bmatrix} = \begin{bmatrix} -21 \\ -9 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ 3 \end{bmatrix} \text{ not an eigenvector}$$

$$\begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 5(6)+(-12)(5) \\ 2(6)+(-5)(5) \end{bmatrix}$$

$$= \begin{bmatrix} 30-60 \\ 12-25 \end{bmatrix} = \begin{bmatrix} -30 \\ -13 \end{bmatrix} \neq \lambda \begin{bmatrix} 6 \\ 5 \end{bmatrix} \text{ not an eigenvector}$$

$$\begin{bmatrix} 5 & -12 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5(2)+(-12)(1) \\ 2(2)+(-5)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 10-12 \\ 4-5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ eigenvector}$$

2. Finding Eigenvalues

By definition, an eigenvalue λ satisfies the equation

$$A\vec{v}_\lambda = \lambda\vec{v}_\lambda.$$

But, rearranging this equation, we see

$$0 = \lambda\vec{v}_\lambda - A\vec{v}_\lambda$$

$$\Rightarrow \quad 0 = (\lambda I - A)\vec{v}_\lambda$$

This means that the matrix $\lambda I - A$ is *singular*, which is to say

$$\det(\lambda I - A) = 0.$$

Key Concept: The **eigenvalues** of a matrix A are the λ such that

$$\det(\lambda I - A) = 0.$$

FACT 75. $\det(\lambda I - A)$ is a polynomial of degree 2 when A is a 2×2 matrix.

EXAMPLES.

2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & -2 \\ -21 & 3 \end{bmatrix}.$$

Solution: First, we determine $\lambda I - A$:

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -21 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -21 & 3 \end{bmatrix} = \begin{bmatrix} \lambda - 4 & 2 \\ 21 & \lambda - 3 \end{bmatrix}.$$

Second, we find $\det(\lambda I - A)$, which will be a polynomial of degree 2. Write

$$\begin{aligned} 0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 2 \\ 21 & \lambda - 3 \end{vmatrix} &= (\lambda - 4)(\lambda - 3) - 2(21) \\ &= \lambda^2 - 7\lambda + 12 - 42 \\ &= \lambda^2 - 7\lambda - 30 \\ &= (\lambda - 10)(\lambda + 3) \end{aligned}$$

Solving $(\lambda - 10)(\lambda + 3) = 0$, we conclude the eigenvalues of A are

$$\lambda = \boxed{-3, 10}.$$

3. Finding Eigenvectors

Once we find the eigenvalues, we can determine the associated eigenvectors. Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector associated to λ. Setup the equation

$$(\lambda I - A) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and then solve for x, y. This always yields a *consistent dependent* system of equations. Recall that solving such a system involves introducing a free parameter, t. The easiest way to do this is to put the augmented matrix

$$\left[\begin{array}{c|c} \lambda I - A & \begin{matrix} 0 \\ 0 \end{matrix} \end{array} \right]$$

into **row-echelon** form.

Key Concept: The **eigenvectors** of A are the solutions to

$$\left[\begin{array}{c|c} \lambda I - A & \begin{matrix} 0 \\ 0 \end{matrix} \end{array} \right]$$

for each eigenvalue λ of A.

Examples.

3. Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 0 & -8 \\ 9 & -17 \end{bmatrix}.$$

Solution: We first need to determine the eigenvalues before we find the corresponding eigenvectors.

Eigenvalues: We find the λ such that

$$\begin{aligned} 0 = \det(\lambda I - A) &= \det\left[\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -8 \\ 9 & -17 \end{bmatrix}\right] \\ &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -8 \\ 9 & -17 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda & 8 \\ -9 & \lambda + 17 \end{vmatrix} \\ &= \lambda(\lambda + 17) - (8)(-9) \\ &= \lambda^2 + 17\lambda + 72 \\ &= (\lambda + 9)(\lambda + 8) \end{aligned}$$

Hence, $\lambda = \boxed{-9, -8}$ are the eigenvalues of the matrix.

Eigenvectors: We need to determine the eigenvectors for both eigenvalues. First, set up the matrix $\lambda I - A$:

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -8 \\ 9 & -17 \end{bmatrix} = \begin{bmatrix} \lambda & 8 \\ -9 & \lambda + 17 \end{bmatrix}.$$

Next, for each of our eigenvalues, we put the following augmented matrix into row-echelon form:

$$\left[\begin{array}{cc|c} \lambda & 8 & 0 \\ -9 & \lambda + 17 & 0 \end{array}\right].$$

$\underline{\lambda = -9}$: Write

$$\left[\begin{array}{cc|c} -9 & 8 & 0 \\ -9 & 8 & 0 \end{array}\right] \xrightarrow{-R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -9 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_1/9 \to R_1} \left[\begin{array}{cc|c} 1 & -8/9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Now, let $y = t$ be the free parameter. Then the solution to this system of equations is

$$x - \frac{8}{9}t = 0 \quad \Longleftrightarrow \quad x = \frac{8}{9}t.$$

Thus, any eigenvector of A associated to $\lambda = -9$ is of the form

$$\begin{bmatrix} (8/9)t \\ t \end{bmatrix}$$

as long as $t \neq 0$. We need just one eigenvector, so choose $t = 9$. We conclude an eigenvector associated to $\lambda = -9$ is

$$\begin{bmatrix} 8 \\ 9 \end{bmatrix}.$$

$\underline{\lambda = -8}$: Write

$$\left[\begin{array}{cc|c} -8 & 8 & 0 \\ -9 & 9 & 0 \end{array}\right] \xrightarrow{-R_1/8 \to R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -9 & 9 & 0 \end{array}\right] \xrightarrow{9R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Let $y = t$ be the free parameter. The solution to the system of equations is then

$$x - t = 0 \quad \Leftrightarrow \quad x = t.$$

Therefore, any eigenvector of A associated to $\lambda = -8$ is of the form

$$\begin{bmatrix} t \\ t \end{bmatrix}$$

for $t \neq 0$. Since we need only one such eigenvector, we choose $t = 1$ and write

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

4. Additional Examples

Examples.

1. Which of the following are eigenvectors of the matrix

$$A = \begin{bmatrix} 14 & -12 \\ 20 & -17 \end{bmatrix}?$$

$$\begin{bmatrix} 5 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Solution: We write

$$\begin{bmatrix} 14 & -12 \\ 20 & -17 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5(14) + 7(-12) \\ 5(20) + 7(-17) \end{bmatrix}$$

$$= \begin{bmatrix} 70 - 84 \\ 100 - 119 \end{bmatrix} = \begin{bmatrix} -14 \\ -19 \end{bmatrix} \neq \lambda \begin{bmatrix} 5 \\ 7 \end{bmatrix} \text{ not an eigenvector}$$

$$\begin{bmatrix} 14 & -12 \\ 20 & -17 \end{bmatrix}\begin{bmatrix} -7 \\ -2 \end{bmatrix} = \begin{bmatrix} 14(-7)+(-12)(-2) \\ 20(-7)+(-17)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -98+24 \\ -140+34 \end{bmatrix} = \begin{bmatrix} -74 \\ -106 \end{bmatrix} \neq \lambda \begin{bmatrix} -7 \\ -2 \end{bmatrix} \text{ not an eigenvector}$$

$$\begin{bmatrix} 14 & -12 \\ 20 & -17 \end{bmatrix}\begin{bmatrix} -4 \\ -5 \end{bmatrix} = \begin{bmatrix} 14(-4)+(-12)(-5) \\ 20(-4)+(-17)(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -56+60 \\ -80+85 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = -1\begin{bmatrix} -4 \\ -5 \end{bmatrix} \text{ eigenvector associated to } r=-1$$

$$\begin{bmatrix} 14 & -12 \\ 20 & -17 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 14(3)+(-12)(4) \\ 20(3)+(-17)(4) \end{bmatrix}$$

$$= \begin{bmatrix} 42-48 \\ 60-68 \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \end{bmatrix} = -2\begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ eigenvector associated to } r=-2$$

2. Find the eigenvalues for the matrix

$$A = \begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix}.$$

Solution: We find the λ such that $\det(\lambda I - A) = 0$. We write

$$\begin{aligned} 0 &= \det(\lambda I - A) \\ &= \det\left[\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix}\right] \\ &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda+4 & 2 \\ -4 & \lambda-2 \end{vmatrix} \\ &= (\lambda+4)(\lambda-2) - (2)(-4) \\ &= \lambda^2 + 2\lambda - 8 + 8 \\ &= \lambda^2 + 2\lambda \\ &= \lambda(\lambda+2) \end{aligned}$$

Setting $\lambda(\lambda+2) = 0$, we conclude the eigenvalues of A are

$$\lambda = \boxed{0, -2}.$$

3. Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 15 & -3 \\ -5 & 1 \end{bmatrix}.$$

Solution: We start by finding the eigenvalues of A.

Eigenvalues: Write

$$\begin{aligned} 0 = \det(\lambda I - A) &= \det \left[\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 15 & -3 \\ -5 & 1 \end{bmatrix} \right] \\ &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 15 & -3 \\ -5 & 1 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda - 15 & 3 \\ 5 & \lambda - 1 \end{vmatrix} \\ &= (\lambda - 15)(\lambda - 1) - (3)(5) \\ &= \lambda^2 - 16\lambda + 15 - 15 \\ &= \lambda^2 - 16\lambda \\ &= \lambda(\lambda - 16) \end{aligned}$$

Setting $0 = \lambda(\lambda - 16)$, the eigenvalues of A are

$$\lambda = \boxed{0, 16}.$$

Eigenvectors: For each λ from above, we put the augmented matrix

$$\left[\begin{array}{cc|c} \lambda - 15 & 3 & 0 \\ 5 & \lambda - 1 & 0 \end{array}\right]$$

into row-echelon form.

$\underline{\lambda = 0}$:

$$\left[\begin{array}{cc|c} -15 & 3 & 0 \\ 5 & -1 & 0 \end{array}\right] \xrightarrow{-R_1/15 \to R_1} \left[\begin{array}{cc|c} 1 & -1/5 & 0 \\ 5 & -1 & 0 \end{array}\right] \xrightarrow{-5R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t$, then

$$x - \frac{1}{5}t = 0 \quad \Leftrightarrow \quad x = \frac{1}{5}t.$$

The eigenvectors associated to $\lambda = 0$ are of the form

$$\begin{bmatrix} (1/5)t \\ t \end{bmatrix}$$

for $t \neq 0$. Choosing one eigenvector, take $t = 5$:

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

$\underline{\lambda = 16}$:

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 5 & 15 & 0 \end{array}\right] \xrightarrow{-5R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t$, then

$$x + 3t = 0 \quad \Leftrightarrow \quad x = -3t.$$

Hence, the eigenvectors associated to $\lambda = 16$ are of the form

$$\begin{bmatrix} -3t \\ t \end{bmatrix}$$

for $t \neq 0$. We need only one eigenvector, so let $t = 1$:

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Lesson 36: Eigenvectors and Eigenvalues (II)

1. Quick Review

EXAMPLES.

1. The matrix

$$A = \begin{bmatrix} -11 & 4 & 6 \\ -8 & 1 & 6 \\ -16 & 4 & 11 \end{bmatrix}$$

has $r = -3$ as one of its eigenvalues. Which of the following is an eigenvector associated to this matrix and eigenvalue?

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Solution: For any of these vectors to be an eigenvector of A associated to $r = -3$, then we must be able to write

$$\begin{bmatrix} -11 & 4 & 6 \\ -8 & 1 & 6 \\ -16 & 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We check this via matrix multiplication:

$$\begin{bmatrix} -11 & 4 & 6 \\ -8 & 1 & 6 \\ -16 & 4 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -11(2) + 4(2) + 6(1) \\ -8(2) + 1(2) + 6(1) \\ -16(2) + 4(2) + 11(1) \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ -8 \\ -13 \end{bmatrix} \text{ not an eigenvector}$$

$$\begin{bmatrix} -11 & 4 & 6 \\ -8 & 1 & 6 \\ -16 & 4 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -11(2) + 4(1) + 6(1) \\ -8(2) + 1(1) + 6(1) \\ -16(2) + 4(1) + 11(1) \end{bmatrix}$$

$$= \begin{bmatrix} -12 \\ -9 \\ -17 \end{bmatrix} \text{ not an eigenvector}$$

$$\begin{bmatrix} -11 & 4 & 6 \\ -8 & 1 & 6 \\ -16 & 4 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -11(2) + 4(1) + 6(2) \\ -8(2) + 1(1) + 6(2) \\ -16(2) + 4(1) + 11(2) \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ -3 \\ -6 \end{bmatrix} = -3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ eigenvector associated to } r = -3$$

$$\begin{bmatrix} -11 & 4 & 6 \\ -8 & 1 & 6 \\ -16 & 4 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -11(1) + 4(3) + 6(2) \\ -8(1) + 1(3) + 6(2) \\ -16(1) + 4(3) + 11(2) \end{bmatrix}$$

$$= \begin{bmatrix} 13 \\ 7 \\ 18 \end{bmatrix} \text{ not an eigenvector}$$

2. Factoring Cubic Polynomials

FACT 76. If A is a 3×3 matrix, then $\det(\lambda I - A)$ is a polynomial of degree 3.

Finding the eigenvalues associated to a 3×3 matrix will require factoring a cubic polynomial (polynomial of degree 3). This can be tricky so we go over a few tips. First, let's go over some basics about polynomials:

- A **root** or a **zero** is a number that makes a polynomial equal to 0
- A cubic polynomial has 3 roots, although some may be repeated
- A polynomial is called **monic** if the coefficient of the highest degree term is **1** (**Ex**: $x^2 + 1$ is monic, $2x^2 + 1$ is not)
- If a polynomial is monic, then all the roots of the polynomial divide the constant term

We use this last fact to factor the polynomials.

Key Assumption: The polynomial has only integer roots.

Method: Let $f(x)$ be a monic polynomial of degree 3. To factor $f(x)$, apply the following:

(1) Write out all the divisors of the constant term

(2) Plug those values into $f(x)$ until you find a root

(3) Use polynomial long division or synthetic division to factor $f(x)$ into a linear term and quadratic term

(4) Factor the quadratic term

Ex 1. Factor the polynomial

$$f(x) = x^3 - 4x^2 + x + 6$$

Step (1): Divisors of 6: $\pm 1, \pm 2, \pm 3, \pm 6$

Step (2):

$\underline{x = 1}$: $(1)^3 - 4(1)^2 + 1 + 6 = 1 - 4 + 1 + 6 \neq 0$ **not a root**

$\underline{x = -1}$: $(-1)^3 - 4(-1)^2 + (-1) + 6 = -1 - 4 - 1 + 6 = 0$ **a root**

Step (3):

$$\begin{array}{r|rrrr} \boxed{-1} & & -1 & 5 & -6 \\ & 1 & -4 & 1 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

which means

$$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$$

Step (4): $x^2 - 5x + 6 = (x - 2)(x - 3)$

We conclude that

$$x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3).$$

EXAMPLES.

2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -9 & 4 & 8 \\ -10 & 5 & 8 \\ -6 & 2 & 7 \end{bmatrix}.$$

<u>Solution</u>: We find the λ that solves

$$\det(\lambda I - A) = 0.$$

Write

$$\det(\lambda I - A) = \det\left[\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -9 & 4 & 8 \\ -10 & 5 & 8 \\ -6 & 2 & 7 \end{bmatrix}\right] = \begin{vmatrix} \lambda + 9 & -4 & -8 \\ 10 & \lambda - 5 & -8 \\ 6 & -2 & \lambda - 7 \end{vmatrix}.$$

We expand along the first row. Recall that the determinant of a 3×3 matrix expanded along the first row is

$$[(1,1)\text{-entry}]C_{11} + [(1,2)\text{-entry}]C_{12} + [(1,3)\text{-entry}]C_{13}.$$

$\boxed{[(1,1)\text{-entry}]C_{11}}$

$$\begin{aligned}(\lambda+9)(-1)^{1+1}\begin{vmatrix}\lambda-5 & -8\\ -2 & \lambda-7\end{vmatrix} &= (\lambda+9)\left[(\lambda-5)(\lambda-7)-(-8)(-2)\right]\\ &= (\lambda+9)\left[\lambda^2-12\lambda+35-16\right]\\ &= (\lambda+9)\left[\lambda^2-12\lambda+19\right]\\ &= \lambda^3-12\lambda^2+19\lambda+9\lambda^2-108\lambda+171\\ &= \lambda^3-3\lambda^2-89\lambda+171\end{aligned}$$

$\boxed{[(1,2)\text{-entry}]C_{12}}$

$$\begin{aligned}(-4)(-1)^{1+2}\begin{vmatrix}10 & -8\\ 6 & \lambda-7\end{vmatrix} &= 4\left[10(\lambda-7)-(-8)(6)\right]\\ &= 4\left[10\lambda-70+48\right]\\ &= 4\left[10\lambda-22\right]\\ &= 40\lambda-88\end{aligned}$$

$\boxed{[(1,3)\text{-entry}]C_{13}}$

$$\begin{aligned}(-8)(-1)^{1+3}\begin{vmatrix}10 & \lambda-5\\ 6 & -2\end{vmatrix} &= -8\left[10(-2)-(\lambda-5)(6)\right]\\ &= -8\left[-20-(6\lambda-30)\right]\\ &= -8\left[-20-6\lambda+30)\right]\\ &= -8\left[-6\lambda+10\right]\\ &= 48\lambda-80\end{aligned}$$

Putting this all together,

$$\begin{aligned}\det(\lambda I-A) &= \lambda^3-3\lambda^2-89\lambda+171+40\lambda-88+48\lambda-80\\ &= \lambda^3-3\lambda^2-89\lambda+40\lambda+48\lambda+171-88-80\\ &= \lambda^3-3\lambda^2-\lambda+3\end{aligned}$$

Now, that we have $\det(\lambda I-A)$, we set this equal to zero and factor to find our eigenvalues.

Let $f(\lambda)=\lambda^3-3\lambda^2-\lambda+3$. We see this is monic and so we go through our steps to factor.

Step (1): Divisors of 3: $\pm1,\pm3$

Step (2): $\lambda = -1$: $(-1)^3 - 3(-1)^2 - (-1) + 3 = -1 - 3 + 1 + 3 = 0$, **a root**

Step (3):

$$\begin{array}{r|rrrr} \boxed{-1} & & -1 & 4 & -3 \\ & 1 & -3 & -1 & 3 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

which means

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = (\lambda + 1)(\lambda^2 - 4\lambda + 3)$$

Step (4): $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

We conclude that

$$f(\lambda) = \lambda^3 - 3\lambda^2 - \lambda + 3 = (\lambda + 1)(\lambda - 1)(\lambda - 3).$$

Therefore, our eigenvalues are

$$\lambda = \boxed{-1, 1, 3}.$$

3. Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} -6 & 6 & -6 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix}.$$

Solution: We find the eigenvalues and then the corresponding eigenvectors.

Eigenvalues: Let

$$\det(\lambda I - A) = \det\left[\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -6 & 6 & -6 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix}\right] = \begin{vmatrix} \lambda + 6 & -6 & 6 \\ 0 & \lambda - 2 & -2 \\ 0 & -3 & \lambda - 3 \end{vmatrix}.$$

We expand along the first column:

$$[(1,1)\text{-entry}]C_{11} + [(2,1)\text{-entry}]C_{21} + [(3,1)\text{-entry}]C_{31}.$$

$\boxed{[(1,1)\text{-entry}]C_{11}}$

$$\begin{aligned} (\lambda + 6)(-1)^{1+1}\begin{vmatrix} \lambda - 2 & -2 \\ -3 & \lambda - 3 \end{vmatrix} &= (\lambda + 6)\left[(\lambda - 2)(\lambda - 3) - (-2)(-3)\right] \\ &= (\lambda + 6)\left[\lambda^2 - 5\lambda + 6 - 6\right] \\ &= (\lambda + 6)(\lambda^2 - 5\lambda) \\ &= \lambda(\lambda + 6)(\lambda - 5) \end{aligned}$$

$\boxed{[(2,1)\text{-entry}]C_{21}}$

$$0(-1)^{2+1}\begin{vmatrix} -6 & 5 \\ -3 & \lambda-3 \end{vmatrix} = 0$$

$\boxed{[(3,1)\text{-entry}]C_{31}}$

$$0(-1)^{3+1}\begin{vmatrix} -6 & 6 \\ \lambda-2 & -2 \end{vmatrix} = 0$$

Thus,

$$0 = \det(\lambda I - A) = \lambda(\lambda+6)(\lambda-5).$$

Our eigenvalues are therefore

$$\lambda = \boxed{-6, 0, 5}.$$

Next, we find the eigenvectors associated to these eigenvalues.

Eigenvectors: We put

$$\left[\begin{array}{ccc|c} \lambda+6 & -6 & 6 & 0 \\ 0 & \lambda-2 & -2 & 0 \\ 0 & -3 & \lambda-3 & 0 \end{array}\right]$$

into row-echelon form for each eigenvalue λ.

$\underline{\lambda = -6}$:

$$\left[\begin{array}{ccc|c} 0 & -6 & 6 & 0 \\ 0 & -8 & -2 & 0 \\ 0 & -3 & -9 & 0 \end{array}\right] \xrightarrow{-R_1/6 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -8 & -2 & 0 \\ 0 & -3 & -9 & 0 \end{array}\right]$$

$$\xrightarrow{8R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & -3 & -9 & 0 \end{array}\right] \xrightarrow{3R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & -12 & 0 \end{array}\right]$$

Now, we see this implies that $y = z = 0$ because $-10y = 0$ and $-12z = 0$. Notice that there is not constraint on x — any x-value will make this system of equations true. So all the eigenvectors associated to $\lambda = -6$ are of the form

$$\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$

for $x \neq 0$. Since we only need one eigenvector, let $x = 1$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$\underline{\lambda = 0}$:

$$\left[\begin{array}{ccc|c} 6 & -6 & 6 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \xrightarrow{R_1/6 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2/2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \xrightarrow{3R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = t$. Then we know that

$$y + t = 0 \quad \iff \quad y = -t.$$

Further, we see

$$x - y + t = 0 \quad \iff \quad x + t + t = 0 \quad \iff \quad x = -2t.$$

All eigenvectors associated $\lambda = 0$ are of the form

$$\begin{bmatrix} -2t \\ -t \\ t \end{bmatrix}$$

for $t \neq 0$. For our answer, we take $t = 1$:

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}.$$

$\underline{\lambda = 5}$:

$$\left[\begin{array}{ccc|c} 11 & -6 & 6 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & -3 & 2 & 0 \end{array}\right] \xrightarrow{R_1/11 \to R_1} \left[\begin{array}{ccc|c} 1 & -6/11 & 6/11 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & -3 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2/3 \to R_2} \left[\begin{array}{ccc|c} 1 & -6/11 & 6/11 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & -3 & 2 & 0 \end{array}\right] \xrightarrow{3R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -6/11 & 6/11 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = t$. We know that

$$y - \frac{2}{3}t = 0 \quad \iff \quad y = \frac{2}{3}t.$$

Hence, we know

$$\begin{aligned} 0 &= x - \frac{6}{11}y + \frac{6}{11}t \\ &= x - \frac{6}{11}\left(\frac{2}{3}\right)t + \frac{6}{11}t \\ &= x - \frac{4}{11}t + \frac{6}{11}t \\ &= x + \frac{2}{11}t \\ \Rightarrow -\frac{2}{11}t &= x \end{aligned}$$

Thus, we conclude eigenvectors associated to $\lambda = 5$ are of the form

$$\begin{bmatrix} -(2/11)t \\ (2/3)t \\ t \end{bmatrix}$$

for $t \neq 0$. We choose our eigenvector by taking $t = 33$, then

$$\begin{bmatrix} -6 \\ 22 \\ 33 \end{bmatrix}.$$

APPENDIX A

Functions

1. General Theory

A basic question that isn't always answered directly in math classes is this: what is a function?

A function is something that takes an input and sends it to an output with the stipulation that identical inputs must go to the same output. The inputs and outputs can be anything we want. For example, suppose we have a list of items from a grocery store:

Items
Gallon of Milk
Frozen Pizza
Mint Chocolate Chip Ice Cream (1 pint)
Avocado
Soda (1 liter bottle)

We'll call this list the **inputs**. Now, let's put together a function which takes each of these inputs and provides their price, which we'll label **outputs**:

Prices
Price(Gallon of Milk)=2.15
Price(Frozen Pizza)=5.74
Price(Mint Chocolate Chip Ice Cream (1 pint))=3.56
Price(Avocado)=Price(Soda (1 liter bottle))=1.37

Price is a function: it sends one item to one price. Note, however, that the prices need not be distinct.

If we took our lists in the other direction (that is, sending prices to items), we would not have a function because the input 1.37 would be sent to two distinct items, the avocado and the soda.

However, this example is unlikely to appear in a math class so let's consider two, more pertinent examples.

Ex 2. Suppose we were given $f(x) = x^2 + 1$ for $0 \leq x \leq 1$. Is this a function? The $0 \leq x \leq 1$ tell us our inputs are real numbers from 0 to 1 and our outputs are some algebraic combination of these inputs (in particular, we are supposed to square the input and then add 1). If we think about it, this is a function because for any input x, we get only one output $x^2 + 1$. If you sketch the graph, this would be an application of the *vertical line test*.

The **notation** $f(x)$ is only meant as shorthand (so we don't have to keep copying down the specifics of the function every time we refer to it). In terms of functions, whenever there are $(\cdot)$, this is informing the reader what inputs are allowed.

Ex 3. Is $f(y) = y^2 + 1$ a function? Well, if you graph it, it **fails** the vertical line test. But the vertical line test *only* applies to functions of x (in the usual xy-plane). This is not a function of x because for $x = 2$, there are two outputs ($y = -1$ and $y = 1$). But as the function is written, the inputs for this are **not** x *but* y. So, while this is not a function of x, it **is a function of** y. The takeaway here is that whether something is a function depends on what inputs we are considering.

2. Polynomials

Polynomials are functions that look like

$$x^3 + x + 1, \quad -37x^{16} + 3x^4 + x^3 - 7x^2 + 12, \quad -4, \quad x + 1.$$

These should be familiar from previous math classes since they are among the most well-behaved functions in mathematics. The following **do not count as polynomials**:

$$x^{3/2} + \sqrt{x+1}, \quad x^3 + 1 + x^{-3}, \quad \frac{x^3 + x + 1}{x^7 + 27}, \quad 2^x + 2.$$

Polynomials always have the variable as the base and the exponents are always non-negative integers (for 3^7, 3 is the base and 7 is the exponent). Note that **all constants** can be thought of as polynomials.

We say the **degree** of a polynomial is the largest exponent. For example,

$$\deg(-7x^3 + 2) = 3, \quad \deg(-3x + 16) = 1, \quad \deg(x^{17} + 4x^2 + x) = 17, \quad \deg(-3) = 0.$$

A **root** or **zero** of a polynomial is an input that makes the polynomial equal to zero. For example, $x = 2$ is a root of $x^2 - 3x + 2$ because

$$2^2 - 3(2) + 2 = 4 - 6 + 2 = 0.$$

The number of roots of a polynomial equals the degree of the polynomial. $x^2 - 3x + 2$ has 2 roots ($x = 1, 2$). However, these roots need not be distinct. The root $x = 2$ appears twice as a root of $x^2 + 4x + 4$.

We say a polynomial is **monic** if the coefficient of the highest degree term is 1. For example, $-3x + 16$ is not monic but $x^{17} + 4x^2 + x$ is monic.

Rational functions are simply polynomials divided by polynomials:

$$\frac{x^3 + x + 1}{x^7 + 27}, \quad -\frac{4}{x}, \quad \frac{x^7 + x^3 + 1}{x^8 + x}.$$

3. Exponential and Logarithmic

An exponential function is a function whose base is fixed and whose exponent changes:

$$e^x, \quad 2^x, \quad 13^{x^2 + 7x + y}.$$

Logarithmic functions are the function inverses of exponential functions. For a some fixed number,

$$\log_a a^x = x \quad \text{and} \quad a^{\log_a x} = x.$$

In this class, we focus on e^x and $\ln x$ since other exponential and logarithmic functions behave in a very similar way.

$\underline{e^x}$: e^x is a common function that might appear to be intimidating. Something to know is that $e \approx 2.71828$ is a **number**. It is not a variable or anything exotic — it is just a number (albeit an important number). e^x simply means we are taking e and multiplying it by itself x times.

e^x is helpful example for exploring different exponent rules, consider:

$$e^{a+b} = e^a e^b \quad \text{and} \quad e^{a-b} = e^a e^{-b}.$$

If something is being *added* or *subtracted* in the exponent, then we can "separate" over the same base. This also applies when the base is a variable:

$$x^{a+b} = x^a x^b \quad \text{and} \quad x^{a-b} = x^a x^{-b}.$$

Moreover,

$$(e^a)^b = e^{ab} = (e^b)^a.$$

This means that when two things are being *multiplied* in the exponent, then the entire function is being raised to a power. Observe that

$$e^{x^2} = e^{x\cdot x} = (e^x)^x$$

is **not** the same as

$$e^{2x} = (e^x)^2.$$

Again, this also applies to the case when the base is a variable:

$$x^{ab} = (x^a)^b.$$

Recall that $e^{-x} = \dfrac{1}{e^x}$ which means

$$\frac{1}{e^{-x}} = \frac{1}{\frac{1}{e^x}} = e^x.$$

Similarly,

$$x^{-n} = \frac{1}{x^n} \quad \text{and} \quad \frac{1}{x^{-n}} = x^n.$$

Observe that

$$\frac{x^{an}}{x^{bn}} = \left(\frac{x^a}{x^b}\right)^n.$$

Finally, note that $e^x > 0$ for any x value.

$\underline{\ln x}$: All logarithmic functions come from $\ln x$ so we don't lose any generality focusing only on $\ln x$. The primary characteristic of $\ln x$ is that it "undoes" e^x, that is,

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$$

This means that $\ln x$ is the function inverse of e^x. $\ln x$ is very useful because of three of its properties:

(1) $a \ln b = \ln b^a$

(2) $\ln(ab) = \ln a + \ln b$

(3) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln x$ is only defined for $x > 0$.

4. Basic Trigonometry

Trigonometric functions relate the angles of right triangles to ratios of the lengths of their sides:

$$\begin{array}{c|c} \cos\theta = \dfrac{\text{adj}}{\text{hyp}} & \sin\theta = \dfrac{\text{opp}}{\text{hyp}} \\ \sec\theta = \dfrac{\text{hyp}}{\text{adj}} & \csc\theta = \dfrac{\text{hyp}}{\text{opp}} \\ \tan\theta = \dfrac{\text{opp}}{\text{adj}} & \cot\theta = \dfrac{\text{adj}}{\text{opp}} \end{array}$$

Further, the following are important trig identities:

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \end{aligned}$$

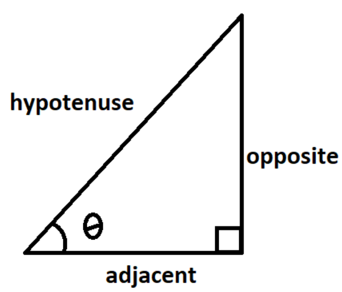


FIGURE 1. Right triangle with angle θ

Right triangles are also used to draw out the unit circle. Recall, on the unit circle, $\cos\theta$ is the x-value for any ordered pair and $\sin\theta$ is the y-value where θ is an angle.

Once we know $\cos\theta$ and $\sin\theta$, we can determine all the other trig functions:

$$\begin{array}{c|c} \sec\theta = \dfrac{1}{\cos\theta} & \tan\theta = \dfrac{\sin\theta}{\cos\theta} \\ \csc\theta = \dfrac{1}{\sin\theta} & \cot\theta = \dfrac{\cos\theta}{\sin\theta} \end{array}$$

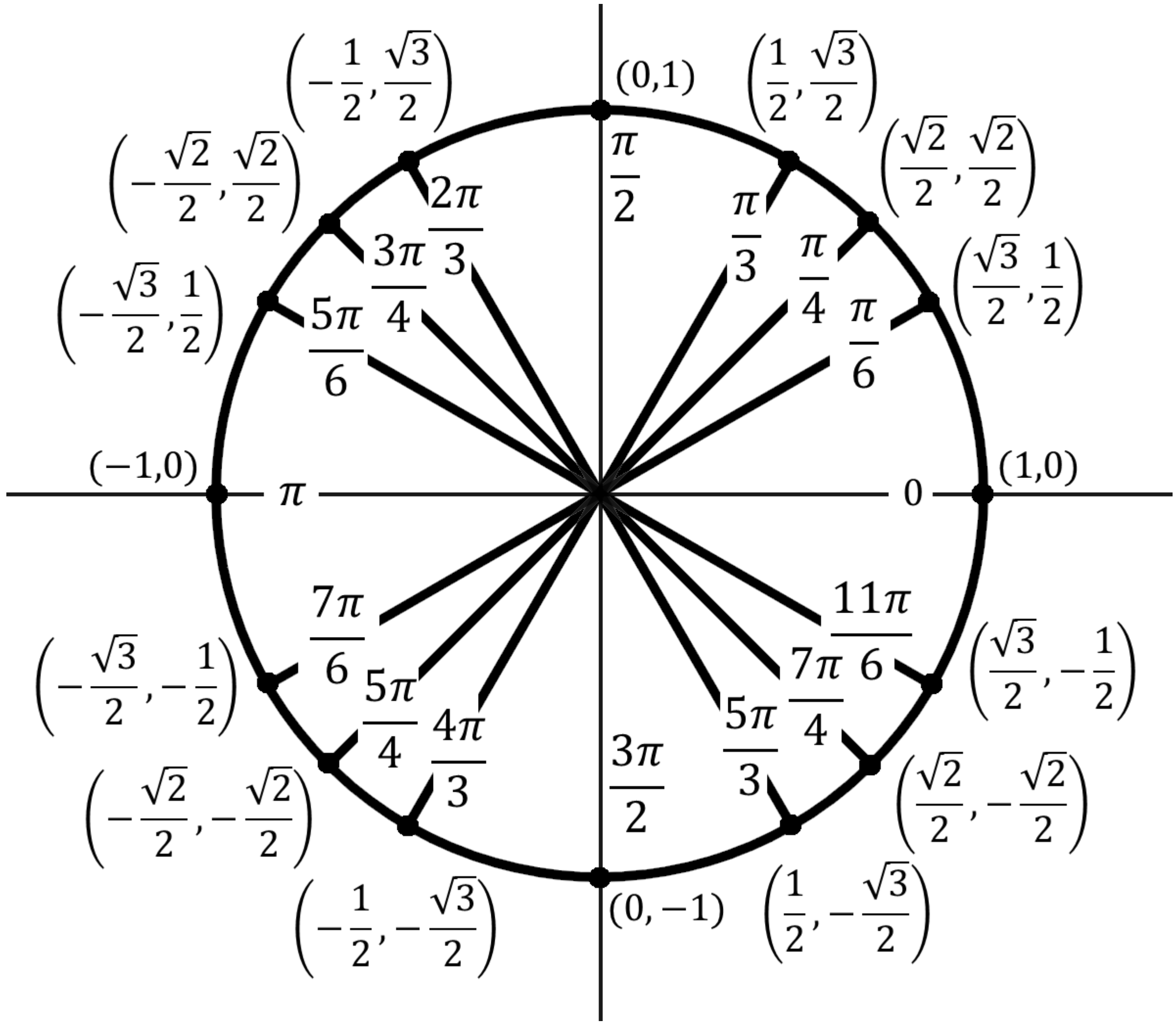


FIGURE 2. Unit Circle

APPENDIX B

Basic Differentiation Table

Let k and n be real numbers (that is, numbers like $-17, \pi, 112.76$, etc). Let $f(x)$, $g(x)$ be functions in the variable x.

Properties of Differentiation	
Addition	$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$
Subtraction	$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$
Constant Multiplication	$\frac{d}{dx}kf(x) = kf'(x)$

Basics	
Polynomials	$\frac{d}{dx}k = 0$ $\frac{d}{dx}x^n\,dx = nx^{n-1}$ for $n \neq 0$
Trig	$\frac{d}{dx}\cos(x) = -\sin(x)$ $\frac{d}{dx}\sin(x) = \cos(x)$ $\frac{d}{dx}\tan(x) = \sec^2(x)$ $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
Exponential	$\frac{d}{dx}e^x = e^x$
Logarithmic	$\frac{d}{dx}\ln(x) = \frac{1}{x}$ when $x > 0$

Rules	
Chain Rule	$\frac{d}{dx}[f(g(x))] = g'(x)f'(g(x))$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$
Quotient Rule	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

APPENDIX C

Basic Integration Table

Let k and n be real numbers (that is, numbers like $-17, \pi, 112.76$, etc). Let $f(x)$, $g(x)$ be functions in the variable x. C stands for an arbitrary constant.

Properties of Integration	
Addition	$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$
Subtraction	$\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$
Constant Multiplication	$\int k f(x)\, dx = k \int f(x)\, dx$

Basics	
Polynomials	$\int 0\, dx = C$ $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C$ for $n \neq -1$ $\int x^{-1}\, dx = \ln\lvert x\rvert + C$
Trig	$\int \cos(x)\, dx = \sin(x) + C$ $\int \sin(x)\, dx = -\cos(x) + C$ $\int \sec^2(x)\, dx = \tan(x) + C$ $\int \sec(x)\tan(x)\, dx = \sec(x) + C$
Exponential	$\int e^x\, dx = e^x + C$

APPENDIX D

Exact VS. Decimal Answers

If Loncapa does not specify a decimal place, your answers are expected to be **exact**. What does that mean? Consider the following example.

Suppose $\dfrac{16}{3}$ is the answer to a question and instead you input 5.333. This will be marked wrong because these are **NOT** the same number. 5.333 is only an **approximation** of $\dfrac{16}{3}$. To see this, write

$$\frac{16}{3} - 5.333 = \frac{16,000}{3,000} - \underbrace{\frac{15,999}{3,000}}_{5.333} = \frac{1}{3,000}$$

which is small certainly, **but it is not 0**.

$\dfrac{16}{3}$ is **exact** because you have all of the number's information. In contrast, a decimal approximation forces you to chop off the end of the number, thus losing information.

Note that not all decimals are rounded. For example, $\dfrac{1}{2} = .5$ because

$$\frac{1}{2} - .5 = \frac{1}{2} - \frac{2}{2}(.5) = \frac{1}{2} - \frac{1}{2} = 0.$$

There is no rounding which occurs when going from $\dfrac{1}{2}$ to .5 which means .5 is **exact**.

The rule of thumb is this: if you need to round, then the answer is **not exact**. In particular, if you plug a number into a calculator and the numbers keep going past the edge of the screen, then the answer you are putting down is going to be rounded.

APPENDIX E

Set Builder Notation

What does

$$\{(x,y) : x \geq 2, y \leq 1\}$$

mean? This just means we are looking at the **set** of ordered pairs (x, y) such that $x \geq 2$ and $y \leq 1$. Whenever we write {something} we mean the *set* of something. What's a set? It's just a collection of things. For example,

$$\{\text{bat}, \text{cat}, \text{mat}\}$$

is a set of 3 things that rhyme and

$$\{\Box, \triangle, \bigcirc, \diamond\}$$

is a set of 4 shapes.

But sometimes, we want to take a set and talk about a smaller set (called a subset). So we could write

$$\{\text{shapes} : \text{the shape has 3 sides}\}.$$

We think of the first part

$$\{\underbrace{\text{shapes}}_{\text{this part}} : \text{the shape has 3 sides}\}$$

as the *type* of thing we are looking at and the second part

$$\{\text{shapes} : \underbrace{\text{the shape has 3 sides}}_{\text{this part}}\}$$

tells us how those first things need to be. Read the ":" as a "such that". This means that

$$\{\text{shapes} : \text{the shape has 3 sides}\} = \text{the collection of shapes such that the shape has 3 sides}$$

This way of describing collections of things is called **set builder notation**.

Let's interpret what the set

$$\{(x,y) : \ln(x+y) \neq 1\}$$

means. Well, we are looking at ordered pairs (x, y) such that $\ln(x + y) \neq 1$. But if we think about this, we know that $\ln e = 1$, which means we only want $x + y = e$. Hence, the two sets

$$\{(x,y) : \ln(x+y) \neq 1\} \quad \text{and} \quad \{(x,y) : x+y \neq e\}$$

are *actually the same set*. Much of the time, there is more than one way to describe a set in a helpful way.

APPENDIX F

$n!$ (n Factorial)

The expression $n!$ (read: n factorial) means

$$n! = n(n-1)\cdots 2 \cdot 1$$

with the convention that $0! = 1$. For example,

$$\begin{aligned}
&1! = 1 \\
&2! = 2 \cdot 1 = 2 \\
&3! = 3 \cdot 2 \cdot 1 = 6 \\
&4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\
&5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\
&6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \\
&\quad\vdots
\end{aligned}$$

and so on in that pattern. In words, you multiply all the numbers between 1 and n including 1 and n. So 12! is the product of *all* the numbers between 1 and 12, **including** 1 and 12.

Example 1.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n!} &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \\
&= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots
\end{aligned}$$

Fact 77. Because of how $n!$ is defined,

$$\frac{n!}{(n-1)!} = n.$$

Example 2.

$$\frac{7!}{6!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7.$$